Comparative study of current profile evolution on the JET tokamak using resistive diffusion and ideal magnetohydrodynamic equilibrium models.

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Abstract

This thesis consists of a study of the magnetic field diffusion equation and magnetic equilibrium reconstructions using the transport analysis code TRANSP and the free-boundary equilibrium code EFIT in the JET tokamak. TRANSP uses the magnetic field diffusion equation to calculate the evolution of the current profile. The thesis includes a description of the TRANSP code, with emphasis on the implementation of current diffusion. The pitch angle predicted by TRANSP is compared with motional Stark effect (MSE) polarimetry data. The accuracy of the TRANSP predictions depends on the validity of the resistivity model employed. The comparison is carried out on a large range of pulses, including cases of current hole plasmas (i.e. plasmas with a region of close to zero current density around the magnetic axis). In particular, the evolution of the pitch angle as the current hole contracts is investigated. The TRANSP output is also compared to Faraday rotation polarimetry, MHD and surface voltage data. In addition to comparing TRANSP and EFIT output for consistency, the results confirm that a neoclassical model of the resistivity is superior to the classical one in modelling the evolution of the current profile in JET. The Hirshmann, NCLASS and Sauter neoclassical models of resistivity and bootstrap current are employed in the study. The invalidity of existing models of the bootstrap current in the current hole region is reflected by the decreased accuracy of the pitch angle predictions in these cases. This thesis also includes an assessment of the interpretative iron model used in the EFIT equilibrium code. The effect that modifications have on the EFIT equilibrium results are evaluated. Finally, the diffusion of the magnetic field in plasmas of non-uniform resistivity is studied. The solution to the magnetic field diffusion equation in simple cases of non-uniform resistivity is derived. It is found that a single quantity, an effective diffusivity, describes the evolution of the magnetic field across the plasma, even though the local resistivity varies. For resistivity profiles based on a wide range of realistic plasma parameters, the magnetic field diffusion equation is solved numerically, leading to a model of effective magnetic diffusivity. TRANSP calculations of the magnetic field evolution are consistent with the model. Separate approaches to finding the effective magnetic diffusivity are suggested, including one based on the analytical solutions.
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Chapter 1

Introduction

1.1 Why nuclear fusion?

The ever increasing energy requirements of the developing world, as the pace of
their industrialisation quickens (see figure 1.1), will place a considerable
strain on the Earth’s finite carbon-based fuel resources over the coming cen-
tury (which currently meets over 80% of the world’s energy needs [2]). Crude
oil production is expected to begin declining due to resource constraints be-
tween 2007 and 2013 [3]. Even if energy consumption were to remain constant
at current levels, proved reserves of petroleum, natural gas and coal would
last for 40, 60 and 200 years respectively [4]. As levels of carbon-based fuel
consumption rises, the consequent increase in CO2 emissions into the at-
mosphere will lead to severe climatic and environmental change [5]. It is
abundantly clear that alternative sources of energy need to be researched
and implemented. A plethora of alternative sources of energy, such as wind,
solar, biomass, hydro and tidal, will contribute increasing amounts of energy
over the coming decades. However, each of these sources of energy suffer
from inherent disadvantages that are likely to limit their applicability.

Nuclear fission energy is capable of satisfying a large percentage of a devel-
oped nation’s energy needs\(^1\). This source of energy is based on the splitting
of a uranium or plutonium ion into lighter elements. The mass difference
\(\Delta m\) between the reactants and products is converted into energy based on
Einstein’s equation \(E = \Delta mc^2\) (\(c\) being the speed of light in vacuum). The
energy released per nuclear reaction is approximately \(10^6\) times greater than
released by a chemical reaction. Thus, while a typical nuclear power sta-

\(^1\)Close to 80% of electricity in France is supplied by its 58 nuclear power stations
1.1 Why nuclear fusion?

Fusion consumes $80 - 100$ tonnes of uranium fuel per annum, a coal power plant producing an equivalent amount of electricity requires $\mathcal{O}10^6$ tonnes of fuel [6]. However, the difficulty of storing the long-lived radioactive products, the finite amount of suitable fuel available and the associated risk of nuclear weapon proliferation are drawbacks which may make nuclear fission power unsuitable to meet the world’s energy demands over the long term. While fuel can be generated through the re-processing of spent fuel and by a new generation of fast breeder reactors, the political will to embark on a nuclear fission power programme has faded in the industrialised world.

![Figure 1.1: World energy consumption by region. The EE/FSU comprise eastern European and former Soviet Union nations. One British thermal unit (Btu) is equivalent to 1.055 kJ. Source: US energy information administration (EIA)](image)

Thus, it is apparent that the world will increasingly need a new type of power plant; one that uses abundant and readily available fuel, that has a minimal negative impact on the environment and that produces a substantial amount of power to meet the demands of a more populous and industrialised world. Such is the promise of nuclear fusion.
1.2 Nuclear fusion

Nuclear fusion is the process of joining two nuclei so that a heavier nucleus is formed. When very light nuclei\(^2\) (such as the isotopes of hydrogen) undergo nuclear fusion the mass difference between the reactants and products, as in the case of nuclear fission, results in a release of energy. This mass difference is due to an increase in the binding energy per nucleon, i.e. the energy required to bind each nucleon to the nucleus, upon the fusion of the two nuclei (see figure 1.2). This energy is imparted as kinetic energy to the products of the reaction. The most energetically favourable reactions involving isotopes of the two lightest elements - deuterium \(D^2\), tritium \(T^3\) and helium-3 are presented below.

\[
\begin{align*}
D^2 + D^2 & \rightarrow T^3 + H^1 + 4.03 \text{MeV} \\ 
D^2 + D^2 & \rightarrow He^3_2 + n^0_0 + 3.27 \text{MeV} \\ 
D^2 + D^2 & \rightarrow He^4_2 + 23.0 \text{MeV} \\ 
D^2 + T^3 & \rightarrow He^4_2 + n^0_0 + 17.6 \text{MeV} \\ 
D^2 + He^3_2 & \rightarrow He^4_2 + H^1 + 18.3 \text{MeV}
\end{align*}
\]

Due to the Coulomb repulsion between two nuclei, extremely high temperatures must be reached for there to be a significant probability that fusion will take place (fusion occurs at energies below the Coulomb barrier due to quantum tunnelling effects). Furthermore, these fast moving ions must be confined for long enough in one place to produce a substantial amount of fusion energy. The cross-section of the D-T reaction (eqn. 1.4) reaches a maximum at a much lower temperature (\(\approx 100 \text{keV}\)) than the other fusion candidates. Each D-T reaction produces an \(\alpha\)-particle and a neutron with kinetic energy 3.5MeV\(^3\) and 14.1MeV respectively. This is the preferred fusion reaction to be exploited in any future fusion reactor.

If a D-T fuel is heated to a temperature of 10keV, sufficient fusion energy would be produced by ions in the high energy tail of the Maxwellian distribution (i.e. thermonuclear fusion). The energy produced by fusion far exceeds the thermal energy of the fuel ions but a large amount of energy is required

\(^{2}\)Super-heavy elements are transiently created via the fusion of heavy nuclei. For example, elements 113 and 115 have been formed from the fusion of \(Ca^{48}\) and \(Am^{243}\).

\(^{3}\)The electron-volt is typically used in plasma physics. \(1 \text{eV} = 1.602 \times 10^{-19} \text{J}\)
1.2 Nuclear fusion

Figure 1.2: Binding energy per nucleon as a function of atomic mass. Source: Hyperphysics, a resource provided by Georgia State University

to achieve thermonuclear conditions. At such high temperatures the fuel is fully ionised and together with the unbound electrons form a quasineutral gas called a plasma. The plasma\(^4\), also known as the fourth state of matter, exhibits complex collective behaviour due to the long-range Coulomb interaction between the charged particles (see section 2.1).

Deuterium makes up a small percentage of the hydrogen found in the world’s oceans and consequently a virtually limitless supply is available (about \(10^{15}\) tons). Only trace amounts of tritium can be found in nature since it is a short-lived radioisotope with a half-life of 12.3 years. Tritium can be bred by lithium-7 with fast neutrons via the fusion reactions

\[
Li^7_3 + n_0^{1\text{(fast)}} \rightarrow He^3_2 + T^3_1 + n_0^{1\text{(slow)}} \tag{1.6}
\]

Tritium is a \(\beta\)-particle emitter and consequently is dangerous only if inhaled, absorbed through the skin or ingested. At present tritium is manufactured as a by-product at particle accelerator facilities. Spallation reactions of the high energy nucleons with oxygen and other heavy molecules in the water cooling system results in the formation of tritiated water (HTO). This is, of course, a very energy consuming way of producing tritium. In a fusion reactor, it is

\(^4\text{from the ancient Greek word for something moulded or formed}\)
envisaged that the fast neutrons produced by D-T reactions (equation 1.4) would be used to breed tritium (equation 1.6) by placing a lithium blanket around the reaction chamber.

The sun produces energy via a self-sustaining chain of nuclear fusion reactions; beginning with the fusion of two protons. In this solar furnace the fuel ions are trapped inside a gravitational potential well, a method of confinement that could not be replicated on Earth. Instead two main confinement methods are pursued - inertial and magnetic. In the former, thermonuclear conditions are reached as a D-T pellet is imploded by laser or X-ray heating. This is the basis of a thermonuclear weapon but harnessing this power for peaceful purposes is a distant prospect. By contrast, great strides towards a fusion power reactor have been made by employing the latter confinement method, that is the magnetic confinement of a high-temperature D-T plasma.

1.3 Magnetic confinement devices

Magnetic confinement devices exploit the tendency of charged particles to gyrate about magnetic field lines due to the Lorentz force. A closed magnetic field geometry is essential to localise the plasma and to prevent contact with material surfaces (such contact causes sputtering of impurities into the plasma and leads to high radiation losses). The desirability of a closed geometry was demonstrated by early cylindrical devices which used magnetic field gradients to confine the plasma (i.e. a magnetic mirror). Confinement was poor due to the ability of a substantial fraction of the charged particles to escape along the open magnetic field lines.

A toroidal magnetic geometry is a natural progression towards a closed geometry from a cylindrical one. Toroidal magnetic confinement devices were first investigated soon after WWII [1]. Initially, the potential military application (as a potential neutron source to make weapons-grade plutonium) of these machines led to research being conducted in secrecy [7]. Early designs were beset by plasma instabilities and lack of confinement. From the assortment of toroidal designs that emerged (which include spheromaks, torsatrons and heliacs), two which gained prominence over the succeeding years are the tokamak and the stellarator. These two devices are distinguished by the methods used to ‘twist’ the magnetic field.

It was found that adding a toroidal magnetic field (thereby creating the
so called toroidal pinch device), using external coils that encircle the toroidal vacuum vessel, brought an improvement in plasma stability. However, the toroidal field alone is unable to provide a counterbalance to the pressure gradient which pushes the plasma radially outwards and so these pinch devices suffered from poor plasma confinement. A poloidal (the poloidal plane is defined by a slice made at any toroidal angle through the plasma) magnetic field, in tandem with a toroidal plasma current, will generate a radially inward \( J \times B \) force and keep plasma in equilibrium (see section 2.3). In a tokamak this poloidal field is created by driving a plasma current in the toroidal direction. The result is a helical field which twists in both the poloidal and toroidal direction. By contrast, in a stellarator the helical field is generated by external field coils that follow complex (non-axisymmetric) geometries. The original stellarator was built by Lyman Spitzer and its increasingly complex successors play an important role in fusion research.

## 1.4 The tokamak

The original proposal\(^5\) for the tokamak design was made by Sakharov and Tamm in 1950 [8]. The word tokamak is derived from the Russian ‘toroidalnaya kamera and magnetnaya katushka’, literally ‘toroidal chamber and magnetic coil’. Valuable pioneering work was carried out at Tokamak-3 (T-3), the largest of a series built at the Kurchatov Institute during the 1960s. The important early achievements included maintaining reasonable plasma equilibrium, reaching temperatures of order 1000eV and the measurement of some plasma parameters such as energy content. These developments stimulated the construction of several other tokamaks around the world.

A tokamak is an axisymmetric toroidal device (also known as a 'torus'), consisting of a vacuum vessel in which the plasma is contained, and an array of magnetic field coils outside the vessel which help to contain and shape the plasma. As mentioned before, toroidal field coils establish a high magnetic field in the toroidal direction. Current is induced in the plasma via a transformer action by ramping up a current in a primary coil which goes around the torus. The plasma current also heats the plasma through ohmic heating.

The magnetic geometry of the plasma is shown in figure 1.3. In this fig-

---

\(^5\)The idea, initially classified, was revealed to the world in 1956 by Igor Kurchatov in a lecture delivered in Harwell, UK.
1.4 The tokamak

Figure 1.3: Schematic of toroidal magnetic geometry. The poloidal $B_{pol}$ and toroidal $B_{tor}$ components combine to give the total magnetic field $B_{total}$. The plasma current $I_p$ and the major and minor radii ($R$ and $a$) are also depicted.

The aspect ratio, i.e. the ratio of the major radius to the minor radius, is substantially greater than unity. Tokamaks conventionally have an aspect ratio between $2 - 3$; those with aspect ratio less than 1.5 are known as spherical tokamaks (e.g. the Mega-Ampère Spherical Tokamak (MAST) and the National Spherical Torus Experiment (NSTX)). A low aspect ratio tokamak promises advantages over the conventional shape in terms of fusion performance and the avoidance of plasma instabilities. As a consequence of the internal force balance between the plasma and magnetic pressure (i.e. magnetohydrodynamic or MHD equilibrium), an infinite set of nested, toroidal magnetic flux surfaces are formed. In the simplest tokamaks, as in figure 1.3, and in the limit of infinite aspect ratio, a cross-section through the plasma would reveal a set of concentric circular magnetic flux surfaces. In general, tokamaks possess a non-circular cross-section to benefit from the associated advantages in plasma stability and confinement.

The plasma is isolated from the vessel wall by employing either a limiter surface or a magnetic divertor. A limiter serves to limit the plasma interaction with a solid surface and is typically made of molybdenum or tungsten capable of withstanding high heat loads [9]. A divertor, on the other hand, avoids confined plasma-surface interactions altogether and allows the exhaust
of plasma particles to be controlled. A poloidal divertor creates a null (or X-point) in the poloidal magnetic field inside of which the magnetic flux surfaces are closed (see figure 1.4). The last closed flux surface in such a configuration is known as the separatrix. Outside the separatrix, open magnetic field lines allow escaping ions and electrons to stream down to the divertor plates.

A key measure of fusion performance is the energy confinement time $\tau_E$ (the ratio of the kinetic energy to the input power). A sudden marked improvement in confinement can occur in divertor plasmas when the heating power exceeds a certain threshold. This is associated with the appearance of a barrier in transport at the plasma edge. This high confinement regime is known as the H-mode and will be exploited in any proposed fusion reactor. Another figure of merit of fusion performance is the parameter plasma $\beta$ (the ratio of the plasma pressure to the magnetic pressure). A high-$\beta$ plasma, at a given magnetic pressure, is one in which the magnetic field efficiently confines a plasma of high temperature and density, thus leading to a high rate of fusion. $\beta$ is also an indicator of cost since, for a given plasma pressure,
a higher value of $\beta$ implies a lower magnetic pressure and, consequently, cheaper toroidal field coils.

The product of the plasma density, ion temperature and energy confinement time, known as the fusion triple product, provides a succinct measure of progress towards a fusion reactor. The Lawson criterion [10] places a lower limit on the triple product of a hypothetical fusion reactor.

$$nT_i \tau_E > 10^{21} \text{keV m}^{-3} \text{s}$$

Although the fusion triple product attained has increased by a factor of $10^5$ over the past 30 years [11], it still falls short of breakeven conditions (i.e. where power produced by fusion reactions equals the input power). Heretofore, experiments indicated an operational limit on the plasma density; the so-called Greenwald density limit [12]. However, in recent years densities above the limit have been achieved [13]. Improvements have also been made, within the past decade, in energy confinement time $\tau_E$ by devising so-called 'advanced tokamak scenarios' (e.g. by creating a transport barrier within the plasma).

A plasma current induced by the transformer action outlined in section 1.4 is incompatible with the steady-state operation required of a reactor. Non-inductive current can be driven by the application of RF waves at the lower hybrid resonant frequency (an intermediate of the ion and electron cyclotron frequencies), by the injection of high-energy neutral beams and by taking advantage of the so-called bootstrap current. Operation at full non-inductive current drive has been accomplished, for 2.7s of a discharge, at the JT-60 tokamak [14].

1.5 JET

The Joint European Torus is, as the name implies, a collaboration between a group of European countries (under the auspices of EURATOM, the European Atomic Energy Community). It is located close to Oxford, UK. It is the largest tokamak in the world and, since the closure of TFTR in 1997, the only one with the capability of operating with a D-T plasma. In 1997 peak fusion power of 16MW was measured when operating with a 50-50 mix of D-T fuel; still a world record for tokamaks. The Q-factor attained, i.e. the
ratio between the fusion power produced to the heating power supplied, was an unsurpassed 0.65.

The main aim of the JET machine is to study how various plasma parameters scale as one approaches reactor conditions. The chief criteria which informed its design was that the machine be capable of achieving thermonuclear conditions and that any $\alpha$-particles produced by fusion reactions could be confined in plasma (necessitating a plasma current of at least 3MA). Since the resistance of the plasma falls as the temperature increases, the plasma cannot be heated to thermonuclear conditions using ohmic heating alone. The plasma is heated to the required temperatures using a combination of neutral beam injection and ion cyclotron resonance heating (ICRH).

D-shaped toroidal field coils (numbering thirty two in all), and consequently a D-shaped vacuum vessel, was decided upon to minimise stress on the coils due to bending forces. The current in the primary coil, which is wound around the central column, is coupled to the plasma using an eight limbed iron transformer core (visible in figure 1.5) within which the vacuum vessel, constructed in octants, sits. The high permeability of the iron core
Table 1.1: JET parameters. Source: EFDA-JET website

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plasma major radius</td>
<td>2.96m</td>
</tr>
<tr>
<td>Plasma minor radius (vertical)</td>
<td>2.10m</td>
</tr>
<tr>
<td>Plasma minor radius (horizontal)</td>
<td>1.25m</td>
</tr>
<tr>
<td>Plasma volume</td>
<td>$153 \text{ m}^3$</td>
</tr>
<tr>
<td>Flat top pulse length</td>
<td>20s</td>
</tr>
<tr>
<td>Volt-seconds to drive plasma current</td>
<td>34 Vs</td>
</tr>
<tr>
<td>Weight of iron core</td>
<td>2800t</td>
</tr>
<tr>
<td>Maximum plasma current</td>
<td>4.5 MA</td>
</tr>
<tr>
<td>Toroidal magnetic field (max)</td>
<td>3.5T</td>
</tr>
<tr>
<td>Neutral beam power (max)</td>
<td>23MW</td>
</tr>
<tr>
<td>ICRF power (max)</td>
<td>$\approx 15\text{MW}$</td>
</tr>
<tr>
<td>Total system input power per pulse</td>
<td>$\approx 500\text{MW}$</td>
</tr>
</tbody>
</table>

ensures efficient coupling of the magnetic flux produced by the primary with the plasma. Outside the toroidal field coils, but inside the iron limbs, six circular poloidal field coils control the plasma shape and position (see figure 1.6).

At JET, a poloidal divertor is used to isolate the plasma from material surfaces and to reach higher confinement modes (see section 1.4). The present gas-box divertor is the latest in a series. Currents flowing in the divertor coils modify the local magnetic field and allow the creation of an X-point. A cryopump removes impurities which result from the interaction of the plasma with the divertor target plates. A series of divertors have been installed, the latest of which using remote handling techniques.

Carrying out such in-vessel modification remotely is necessary following D-T operation when neutron activation of the vessel rules out human access. A force-reflecting servo-manipulator duplicates the movements of an operator in a control room. A 10m long articulated boom with 18 degrees of freedom can reach any point in the vessel within 10mm [15]. A variety of tasks can be carried by the manipulator such as the removal and installation of tiles. The demonstration of this technology is crucial; in the next step machine the neutron flux, from structures surrounding the plasma, will be an order of magnitude higher than it is at JET [16].

Another reactor-relevant technology pioneered at JET is operation with tritium. During D-T operation, tritium is injected into the plasma via the neutral beam heating system or gas puffing. For example, during the 1997 JET operation with D-T plasmas, 35g of tritium were introduced into the
torus, mainly by gas puffing. Over this period a total tritium inventory of 11.5g accumulated in the vessel [17]. This tritium must be thoroughly removed from the vacuum vessel and injection system. The tritium is collected, along with deuterium, using a cryogenic pump and then transferred to uranium beds where it is held in the form of uranium hydrides. At a later stage the hydrogen isotopes are released by heating the uranium and separated using gas chromatography [1]. Tritium operation at JET has been carried out without incident and with very low exposure to personnel [18].

It is clearly advantageous, from the point of view of avoiding instabilities and the precise tailoring of plasma discharges, to have real time control over plasma parameters. In the early years of JET operation, one of the few parameters which could be controlled in real time was the plasma position [19]. Magnetic sensor data and a fast equilibrium code quickly find the last closed flux surface of the plasma. The currents in the poloidal field coils (the actuator), which control the plasmas position and shape, are adjusted
1.6 The future

JET has made possible the research of many physics and technological issues pertinent to a future fusion reactor. It has also shown how international collaboration can work smoothly in running a large-scale fusion experiment. However, it was never designed to produce an excess of fusion power over heating power ($Q > 1$). To achieve a higher fusion amplification factor a new, and considerably larger, machine will have to be built. ITER\(^6\), the international thermonuclear experimental reactor, is the planned next step device. In terms of cost, it will be the second biggest international science project the world has seen (after the international space station) involving China, the E.U., Japan, Russia, South Korea and the United States. The main aim of ITER will be to achieve $Q = 10$ in inductively driven plasmas (pulsed) and $Q = 5$ when driving the current non-inductively (steady-state). It is predicted that ITER will produce fusion power of about 0.5GW. The physics basis for such predictions of ITER performance is based on data from tokamaks worldwide [21]. Running parallel to the ITER project, the International Fusion Materials Irradiation Facility (IFMIF) will develop materials suitable for use in the high neutron flux environment that will be found in a future fusion reactor.

At the time of writing, negotiations regarding the location of ITER and IFMIF are ongoing. JET is being upgraded (notably with an ITER-like ICRF system [22]) to enable it to operate closer to the ITER parameter space.

\(^{6}\text{iter is Latin for 'the way'}\)
1.7 Tokamak plasma current profile

In this section, the topic with which this thesis is concerned is introduced. As outlined in section 1.3, a current is induced in the plasma which, via the Lorentz force, provides a counterbalance to the pressure gradient. Due to the axisymmetry of the tokamak, it may be assumed that the current (and pressure) varies only in the poloidal plane. Furthermore, it may be shown that both the current density and the pressure are functions of the magnetic flux. If measurements of the magnetic field and flux were made inside the plasma, then the current profile could be determined directly. Unfortunately, the high temperature of the plasma preclude the insertion of magnetic sensors in its interior.

Knowledge of the plasma current profile is of pivotal importance in tokamak plasma physics [23] (both from a theoretical and operational point of view). This is particularly so when operating in negative central magnetic shear scenario, one of three 'advanced tokamak' modes [24], wherein the current profile is optimised in controlled stationary conditions [25]. In this scenario, control of the current profile potentially allows an internal transport barrier, beneficial to confinement, to be triggered and subsequently maintained. The current profile is also controlled to avoid certain MHD events. Furthermore, determination of the current profile is important in the study of so-called current holes, a region of zero current density around the plasma magnetic axis [26]. In short, the current profile has major bearing on the equilibrium, stability and transport properties of a tokamak plasma.

The current profile can be determined by solving the MHD equilibrium equation. Assuming axisymmetry this equation (known as the Grad-Shafranov equation, see section 2.3.1) can be expressed as a differential equation whose solution yields the current density profile as a function of poloidal flux. Using numerical techniques to solve this equation, the plasma current profile and flux surface geometry can be reconstructed. Magnetic sensor, polarimetric and kinetic pressure profile data may be employed as constraints on the equilibrium solution. EFIT [27] is a MHD equilibrium code in routine use at JET. The code solves the equilibrium at a series of time points throughout every plasma discharge. With internal constraints, such as polarimetry measurements, the current profile can be reconstructed by EFIT with some confidence. In JET, the current profile reconstruction is complicated by the presence of unknown currents induced in the iron transformer.
core.

A second method to solve the current profile is to use the magnetic field diffusion equation. This equation dictates how irregularities in the magnetic field will resistively diffuse away. If an initial magnetic configuration is prescribed, and appropriate boundary conditions supplied, the current profile can be evolved in time over the plasma discharge. The evolution of the current profile will depend, amongst other things, on the electrical resistivity of the plasma. Due to non-uniformities in the magnetic field a fraction of charged particles will become trapped in the outboard side of the plasma and will be unable to carry current in response to an applied electric field. This effect is taken into account in so-called neoclassical models of the resistivity. The neoclassical resistivity can be several times larger than the classical calculation. The transport analysis code TRANSP provides a comprehensive time dependent analysis of tokamak data. It uses the magnetic field diffusion equation and a neoclassical model of the resistivity to evolve the current profile.

1.8 Thesis outline

This thesis is, as the title suggests, concerned with the comparison of the current profile obtained on JET using resistive diffusion (as calculated by TRANSP) and that reconstructed by the MHD equilibrium code EFIT. The thesis also investigates the effect that the iron core currents have on the EFIT equilibrium solution. The TRANSP and EFIT codes, and particularly their respective methods of determining the current profile, are reviewed.

- Chapter 2 provides the background physics pertaining to the main topics of this thesis. This includes a review of MHD theory, leading to the Grad-Shafranov equation. The magnetic field diffusion equation is also derived, and an analytical solution in the case of cylindrical plasmas with uniform resistivity is derived.

- Chapter 3 describes the EFIT code and investigates the iron model, i.e. the model which simulates the currents in the iron core for the purposes of the EFIT equilibrium reconstruction.

- Chapter 4 reviews TRANSP and describes how current diffusion is implemented in the code. The magnetic field diffusion equation, in
the case of plasmas with non-uniform resistivity, is solved numerically using Mathematica. Some analytic solutions are also found. A model of effective magnetic diffusivity is developed. The results of this analysis are compared to the TRANSP calculations.

- Chapter 5 compares the TRANSP and EFIT current profiles. The magnetic pitch angle (related to the ratio of the poloidal and toroidal magnetic field) is measured by motional Stark effect (MSE) polarimetry. The TRANSP prediction of the pitch angle is compared to this data when using classical and various neoclassical models of resistivity. The EFIT current profile can be calculated using the MSE data as internal constraints. Thus the consistency of the TRANSP and EFIT calculations can be ascertained. In a similar fashion, the TRANSP prediction of the faraday rotation angle measured by polarimetry is compared with data. In addition, the effect of including the TRANSP calculated total pressure profile in EFIT is studied.
Chapter 2

Background

2.1 Fundamental plasma description

A plasma is a quasineutral gas of charged and neutral particles which exhibit collective behaviour [28]. In saying that the plasma is quasineutral we mean that the number of ions and electrons are approximately equal but that electromagnetic forces still exist there. The ions may be fully or partially stripped of their electrons but in the high temperature plasmas common to fusion research the ions are fully stripped. The plasma exhibits collective behaviour due to the long-range Coulomb interaction between charged particles. This long-range interaction means that the behaviour of the plasma on a macroscopic level markedly differs from that of a neutral gas.

An ensemble of ions and electrons must satisfy three conditions to be considered a plasma. These conditions may be expressed as follows,

\[
\frac{\lambda_D}{L} \ll 1 \quad (2.1)
\]

\[
N_D \gg 1 \quad (2.2)
\]

\[
\frac{\omega_p \tau_c}{1} > 1 \quad (2.3)
\]

where \(\lambda_D\) is the Debye length, \(L\) is the length scale of the plasma, \(N_D\) is the number of particles in a Debye sphere, \(\omega\) is the frequency of plasma oscillations and \(\tau_c\) is the mean time between collisions in the plasma. The Debye length is the distance over which the electric field due to a charged particle exerts an influence on its surroundings. A process called Debye shielding ensures that over this distance the electric fields of the individual charged particles cancel each other out. Over distance greater than \(\lambda_D\) the plasma
can be said to be electrically neutral. Thus, in order to ensure the plasma is effectively neutral the Debye length should be much less than the length scale of the plasma (condition 2.1 above). Furthermore, to ensure effective Debye shielding the plasma is required to be sufficiently dense (condition 2.2).

In addition, it is required that the collective effects peculiar to a plasma occur on a time scale short enough that they are not drowned out by collisions. The fundamental collective effect is that of plasma oscillations. This effect is due to displaced electrons subject to a restoring electric field executing simple harmonic motion about the centre of mass of the ions. This motion occurs at the plasma frequency \( \omega_p \)

\[
\omega_p = \sqrt{\frac{n_e e^2}{\epsilon_0 m_e}}
\]  

(2.4)

where \( n_e, e, m_e, \epsilon_0 \) are the electron density, charge, mass and the permittivity of free space respectively. Condition 2.3 simply states that the plasma frequency should exceed the collision frequency.

The study of the physics of plasmas can be approached, broadly, in three ways. The simplest approach is to study single-particle orbits in fixed fields. In a uniform magnetic field \( B \), the charged particle gyrates around the magnetic field line (the radius of this gyration being the Larmor radius \( \rho_L \), the point at the centre of the gyration which moves with the particle is known as the guiding centre). By contrast, the component of the particle’s velocity parallel to the magnetic field line is unaffected. When non-uniformities in the magnetic field or an electric field is introduced, so-called particle drifts result. For example, the addition of a uniform electric field \( E \) results in the guiding centre of the particle drifting in the \( E \times B \) direction. This analysis also yields the three adiabatic invariants of the particle motion - the magnetic moment \( \mu_m \), the longitudinal invariant \( J \) and the total magnetic flux enclosed by the trajectory of the particle. Single particle theory, though useful, is limited as the effects of collisions with other particles is neglected and the influence of the particle itself on its surroundings is not treated in a self-consistent manner.

The second approach is plasma kinetic theory, adopted from the kinetic theory of gases. In this theory the velocity \( \mathbf{v}' \) and displacement \( \mathbf{r} \) of all particles is treated, via the distribution function \( f(\mathbf{r}, \mathbf{v}', t) \), in a statistical fashion. The Vlasov equation describes the evolution of the distribution
function of a collisionless plasma while the Fokker-Planck equation, below, includes a collisional term on the right hand side.

\[
\frac{\partial f}{\partial t} + \mathbf{v}' \frac{\partial f}{\partial \mathbf{r}} + \frac{q_i}{m_i} (\mathbf{E} + \mathbf{v}' \times \mathbf{B}) \frac{\partial f}{\partial \mathbf{v}'} = (\frac{\partial f}{\partial t})_{\text{col}}
\]  

(2.5)

where \( q_i \) and \( m_i \) are the particle charge and mass of a particular plasma species. Other forms of the Fokker-Planck equation are obtained by averaging out the fast gyration about the magnetic field (i.e. the Larmor motion). The drift kinetic equation and the gyro-kinetic equation are important examples used in the study of certain plasma instabilities [9]. Kinetic theory is used in tokamak plasmas to calculate the neutral beam slowing down time and particle collision times. In addition, the Braginskii equations calculate transport processes (such as heat flux) from collisional kinetic theory.

A plasma differs from a normal fluid in its relatively low rate of collisions and the presence of collective effects. However, a fluid approximation, the third approach to a description of plasmas, works surprisingly well. This may be due to the magnetic field in a plasma limiting the transverse velocity (i.e. \( v_{\perp} \)) in a similar way to collisions limiting the velocity in a fluid. The fluid equations are obtained by integrating moments of the Fokker-Planck equation over velocity space. Each particle species (electrons and various types of ions) will have its own system of fluid equations, the equation of continuity and of motion being

\[
\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}_i) = 0
\]  

(2.6)

\[
n_i m_i (\frac{\partial \mathbf{v}_i}{\partial t} + \mathbf{v}_i \cdot \nabla \cdot \mathbf{v}_i) = -\nabla p_i + q_i n_i (\mathbf{E} + \mathbf{v}_i \times \mathbf{B})
\]  

(2.7)

where \( n_i, m_i, v_i \) and \( p_i \) are the density, mass, fluid velocity and scalar pressure of the i-th species. The equation of motion is similar to the Navier-Stokes equation in hydrodynamics except that, in the former, collision terms have been dropped and, in addition, there is an electromagnetic term.

Magnetohydrodynamics (MHD) theory simplifies the analysis still further by assuming that the plasma is composed of a single fluid. The MHD equation of continuity and of motion are derived from the multi-fluid model by summing over the plasma species and also incorporate Maxwell’s equations. The MHD model is described in detail in the following section.
2.2 MHD plasma description

The theory of magnetohydrodynamics treats the plasma as a single fluid in a magnetic field. The equations of MHD combine Maxwell’s equations with those of fluid mechanics. This set of equations is valid over time scales longer than the collisional time $\tau_c$ to ensure the distribution functions of the particle species are locally Maxwellian. Furthermore, it is required that the characteristic length scale be much greater than the ion Larmor radius so that electron diamagnetic and Hall effects may be neglected in the electron momentum equation. Finally, the MHD model assumes that the ions and electrons in the plasma have identical pressures, which together with quasineutrality, amounts to an assumption that the temperatures of the two species equate.

The equations of MHD are listed below. For elegance the convective derivative\(^1\) is employed. The following notation is used: magnetic field $\mathbf{B}$, plasma pressure $p$, plasma current density $\mathbf{J}$, plasma density $\rho$, plasma resistivity $\eta$ and the electric field $\mathbf{E}$.

\[
\begin{align*}
\frac{d\rho}{dt} &= -\rho \nabla \cdot \mathbf{v} \\
\rho \frac{d\mathbf{v}}{dt} &= \mathbf{J} \times \mathbf{B} - \nabla p \\
\frac{dp}{dt} &= -\gamma p \nabla \cdot \mathbf{v} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\
\mathbf{E} + \mathbf{v} \times \mathbf{B} &= \eta \mathbf{J}
\end{align*}
\]

The equations of continuity and of motion (eqn. 2.8 and eqn. 2.9) are obtained from the fluid equations (eqn. 2.6 and eqn. 2.7) by subsuming the plasma species into one fluid. Assuming adiabatic evolution an equation of the pressure can be derived (eqn. 2.10). Since the displacement current is neglected, the current density is given by Amperé’s law (eqn. 2.11). The rate of change of the magnetic field is given by Faraday’s law (eqn. 2.12). To close this system of equations Ohm’s law is also included (eqn. 2.13).

\(^1\)The derivative with respect to a moving coordinate system defined as $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$
In the case of ideal MHD, in which the plasma is assumed to be perfectly conducting, the right hand side of the last equation is zero.

The resistivity of high temperature plasmas (with temperatures in the keV range) is extremely low, in fact at 1 keV it is similar to that of copper at room temperature ($\eta \approx 10^{-8} \, \Omega m$). Thus, over appropriate spatial and temporal scales, the ideal MHD model may be applied with some justification. One consequence of ideal MHD theory is the concept of 'flux freezing'. Consider the flux through some area $S$ in an ideally conducting plasma bounded by a closed loop $C$.

$$\Phi = \int_S B \, dS$$  \hspace{1cm} (2.14)

The loop moves with the plasma at the fluid velocity $v$. In general, the magnetic flux through the loop varies due its movement through the non-uniform magnetic field and due to the time derivative of the field itself. If $dl$ is a line element of loop $C$ then the area it sweeps out in time $dt$ is $v \times dl$ and the flux crossing this area may be written $B \times v \cdot dl$. The total flux derivative is

$$\frac{d\Phi}{dt} = \int_S \frac{\partial B}{\partial t} \, dS + \oint_C B \times v \cdot dl$$  \hspace{1cm} (2.15)

Applying Faraday’s equation and Stoke’s theorem to the terms on the r.h.s allow equation 2.15 to be rewritten

$$\frac{d\Phi}{dt} = -\int_S \nabla \times (E + v \times B) \, dS$$  \hspace{1cm} (2.16)

Substituting the ideal MHD (i.e. $\eta = 0$) form of equation 2.13 into the above expression will immediately lead to the result that the magnetic flux through any loop moving with the plasma is invariant. In other words the magnetic field lines are 'frozen' into the plasma. If each magnetic field line that constitutes the magnetic geometry of the plasma is considered to be an infinitely thin flux tube, this result implies the invariance of magnetic topology. A finite resistivity is required to admit topological changes.

Though the ideal MHD model is a greatly simplified picture of the plasma, it is nonetheless useful in describing plasma instabilities and in the study of plasma equilibrium. The latter application will be examined in the following section.
2.3 MHD equilibrium in toroidal geometry

In a tokamak it is desired that the plasma be in stationary equilibrium. This is equivalent to setting $v = 0$ in the MHD equations\(^2\). The following constitute the set of magnetostatic equations

$$J \times B = \nabla p \quad (2.17)$$

$$\nabla \times B = \mu_0 J \quad (2.18)$$

$$\nabla \cdot B = 0 \quad (2.19)$$

These equations apply over the plasma volume $\Omega_{pl}$; beyond the plasma-vacuum boundary the plasma current density is zero. Since $\nabla \cdot (\nabla \times B) = 0$ it follows from equation 2.18 that $\nabla \cdot J = 0$. This indicates the non-existence of sources of sinks of currents, consistent with the underlying assumption of charge neutrality. From equation 2.17 it is apparent that, in the case of ideal

---

\(^2\)Non-stationary equilibria exist but the kinetic energy due to $v \neq 0$ can be tapped by instabilities.
MHD equilibrium, there is no pressure gradient \( (\mathbf{B} \cdot \nabla p = 0) \) along the magnetic field lines and no component of the current density \( (\mathbf{J} \cdot \nabla p = 0) \) in the direction of the pressure gradient. In other words, the magnetic field lines and the lines of current density lie on magnetic surfaces which correspond to surfaces of constant pressure. The fields \( \mathbf{B} \) and \( \mathbf{J} \) should be non-zero everywhere on the surface in order to maintain force balance. From Poincaré’s 'hairy ball theorem’ it follows that the only smooth topology that will permit an everywhere nonzero tangent vector field is the torus (due to the torus having an Euler Characteristic \( \chi = 0 \)).

The tokamak equilibrium which is implied by the equations of ideal MHD consists of a series of nested toroidal magnetic surfaces, known as flux surfaces (see figure 2.1). Force balance is maintained on each surface via equation 2.17. The limit of the vanishingly small series of toroids is known as the magnetic axis.

![Figure 2.2: The helical trajectory of a magnetic field line on a magnetic flux surface. The change in poloidal angle \( \partial \theta \) after one toroidal rotation is shown. The axes of the cylindrical co-ordinate system \((R, \phi, Z)\) are also shown.](image)

A magnetic field line, \( \mathbf{B} \), will traverse a particular toroidal flux surface in a helical fashion (see section 1.3) until either it eventually closes on itself or until it fills the whole surface. In the former case, the surface is described as being rational since the field line will connect with itself after performing \( m \) toroidal and \( n \) poloidal transits. In the latter case, the surface is an ergodic one. A convenient parameter used to quantify the helicity of a magnetic field
line is the rotational transform \( \iota \), defined as the change in poloidal angle of the field line per toroidal rotation (see figure 2.2). The rotational transform per unit toroidal angle, known as iota-bar \( (\iota) \), is also sometimes used. More widely used, in tokamak research, is the inverse of \( \iota \) known as the safety factor \( q \) due to its importance in stability calculations. On a rational surface \( q \) may be defined as follows

\[
q = \frac{1}{\iota} = \frac{2\pi}{\iota} = \frac{m}{n}
\]  
(2.20)

### 2.3.1 Grad-Shafranov equation

In an axisymmetric plasma, the magnetic field varies only in the poloidal plane and it is convenient to introduce a poloidal magnetic flux function \( \psi \) which is proportional to the poloidal flux within each flux surface. It makes sense to adopt a cylindrical coordinate system \((R, \phi, Z)\) when dealing with axisymmetric toroidal plasmas (shown in figure 2.2). Using a cylindrical coordinate system, the magnetic field in an axisymmetric plasma may be expressed as

\[
B = \frac{1}{R} \nabla \psi \times \hat{\phi} + B_\phi \hat{\phi}
\]  
(2.21)

where \( \hat{\phi} \) is the toroidal unit vector. Substituting this expression for the magnetic field into Ampère’s law (eqn. 2.18) one has (setting all \( \phi \)-derivatives to zero due to the assumption of axisymmetry)

\[
\mu_0 J = \frac{1}{R} \nabla \left( \mu_0 F \right) \times \hat{\phi} - \frac{1}{R} \nabla \left( \frac{1}{R} \nabla \right) \psi \hat{\phi}
\]  
(2.22)

Introducing a current flux function \( F = \frac{R B_\phi}{\mu_0} \) and the Laplacian operator \( \Delta^* \equiv R^2 \nabla \cdot \left( \frac{1}{R^2} \nabla \right) = R \frac{\partial}{\partial R} (\frac{1}{R} \frac{\partial}{\partial R}) + \frac{\partial^2}{\partial Z^2} \) allows equation 2.22 to be rewritten in the following more compact form

\[
\mu_0 J = \frac{1}{R} \nabla (\mu_0 F) \times \hat{\phi} - \frac{1}{R} \Delta^* \psi \hat{\phi}
\]  
(2.23)

where \( \Delta^* \equiv R^2 \nabla \cdot \left( \frac{1}{R^2} \nabla \right) \) is the Laplacian operator. It can readily be shown that the pressure profile \( p \) and the current flux profile \( F \) are functions of the poloidal flux function\(^3\). By considering the radial component of the force

\(^3\)‘flux functions’ are those which depend only on the poloidal flux
balance equation (equation 2.17)

\[
\frac{\partial p}{\partial R} = (\mathbf{J} \times \mathbf{B}) \cdot \hat{R}
\]  

(2.24)

With the help of equations 2.21, 2.23, and noting that \( p \) and \( F \) are flux functions, one may write

\[
\frac{dp}{d\psi} \frac{\partial \psi}{\partial R} = \frac{1}{R} \frac{dF(\psi)}{d\psi} \frac{\partial \psi}{\partial R} \frac{\mu_0 F(\psi)}{R} - \frac{1}{\mu_0 R} \Delta^* \psi \frac{1}{R} \frac{\partial \psi}{\partial R}
\]  

(2.25)

With the manipulation of the above expression the following non-linear second order elliptic partial differential equation can be written

\[
\Delta^* \psi = -\mu_0 R^2 p'(\psi) - \mu^2 F(\psi) F'(\psi)
\]  

(2.26)

where \( p' \) and \( F' \) refer to the gradients of those functions with respect to \( \psi \).

Equation 2.26 is known as the Grad-Shafranov equation [29], [30], [31] and must be solved numerically due to the dependency of the source profiles, \( p(\psi) \) and \( f(\psi) \), on the solution \( \psi(R,Z) \). Solving this equation by various means forms the kernel of any ideal MHD equilibrium code. The preceding derivation is mainly taken from [32].

### 2.4 Diffusion of magnetic field

The flux freezing effect described in section 2.2 applies to plasmas of zero resistivity, or when considering time scales in which the effects of finite resistivity can be ignored. When resistivity is included in the equations of MHD, then restrictions on the magnetic topology of the plasma imposed by flux freezing are lifted and diffusion of the magnetic flux will take place. In order to derive the equation which governs the evolution of the magnetic field Faraday’s law, Ampère’s law and Ohm’s law are required.

\[
\nabla \times \mathbf{E} = -\frac{\delta \mathbf{B}}{\delta t}
\]  

(2.27)

\[
\mathbf{J} = \frac{\nabla \times \mathbf{B}}{\mu_0}
\]  

(2.28)

\[
\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta(\mathbf{J} - \mathbf{J}_{NI})
\]  

(2.29)
2.4 Diffusion of magnetic field

Since, typically, a fraction of the total plasma current in a tokamak plasma is driven non-inductively, this should be subtracted from the total current in Ohm’s law (represented by $J_{NI}$). Substituting the expression for $J$ provided by Ampère’s law into Ohm’s law yields the following expression

$$E = \frac{\eta}{\mu_0} \nabla \times B - \eta J_{NI} - v \times B \quad (2.30)$$

This expression for the electric field can then be substituted into Faraday’s law. Noting that, in general, the resistivity is non-uniform across the plasma, employing the vector identity $\nabla \times (\nabla \times B) = \nabla \cdot (\nabla B) - \nabla^2 B$ and invoking the apparent non-existence of magnetic monopoles one has

$$\frac{\partial B}{\partial t} = \frac{\eta}{\mu_0} \nabla^2 B - \nabla \left( \frac{\eta}{\mu_0} \right) \times (\nabla \times B) + \nabla \times (\eta J_{NI}) + \nabla \times (v \times B) \quad (2.31)$$

This is a generalised form of the magnetic induction equation. The first term on the right hand side describes the resistive diffusion of the magnetic field through the plasma; the last term is due to the convection of the magnetic field with the plasma. The ratio of these two terms is called the magnetic Reynolds number $R_m$.

$$R_m = \frac{\left| \nabla \times (v \times B) \right|}{\left| \frac{\eta}{\mu_0} \nabla^2 B \right|} \approx \frac{\mu_0 V L}{\eta} \quad (2.32)$$

where $V$ is a typical plasma velocity and $L$ is the the length scale of the plasma. In plasmas where the convective term dominates ($R_m >> 1$), ideal MHD theory applies (see section 2.2). In the diffusive limit ($R_m << 1$), and ignoring the terms on the right hand side due to the resistivity gradient and the non-inductive current density, equation 2.31 becomes

$$\frac{\partial B}{\partial t} = \lambda_m \nabla^2 B \quad (2.33)$$

where $\lambda_m = \frac{\eta}{\mu_0}$ is the magnetic diffusivity ($\lambda_m$ has units $m^2 s^{-1}$). The resistive diffusion time $\tau_m$ is the time taken for the magnetic field to become spatially uniform. It is approximated by $L^2/\lambda_m$ where $L$ is the length scale of the plasma. Equation 2.33 is known as the magnetic field diffusion equation (MFDE) and is analogous to the heat conduction equation.
2.4 Diffusion of magnetic field

2.4.1 Magnetic field diffusion example

In order to find an analytical solution to eqn. 2.33 consider the case where the spatial variation in the magnetic field occurs in one dimension only. Perhaps the simplest non-uniformity to consider is that of the magnetic field produced by an infinite current sheet at \( x = 0 \). The derivation presented in this section is in large part due to A. Hood [33]. In this case, the polarity of the magnetic field reverses at the current sheet, its only component is along \( z \)-axis \( \mathbf{B} = B(x)\hat{\mathbf{z}} \), and its magnitude initially on either side of the origin is constant. It is also of interest to include the effect of a resistivity profile which varies in the \( x \)-direction only \( \eta(x) \). Since the variation in the magnetic field and the resistivity is in the \( x \) direction only, and assuming the non-inductive current and convective terms are zero, then equation 2.31 can be written

\[
\frac{\partial \mathbf{B}}{\partial t} = \lambda_m \frac{\partial^2 \mathbf{B}}{\partial x^2} + \frac{1}{\mu_0} \frac{\partial \eta}{\partial x} \frac{\partial \mathbf{B}}{\partial x} \tag{2.34}
\]

The initial magnetic field is given by \( \mathbf{B} = \mathbf{B}_0(x)\hat{\mathbf{z}} \) where \( \mathbf{B}_0(x) = B_0, x > 0 \) and \( \mathbf{B}_0(x) = -B_0, x < 0 \). Equation 2.34 is a partial differential equation (PDE) which we seek to transform to an ordinary one (ODE). The solution for the magnetic field in the equation above will depend on the choice of resistivity function. In order to proceed, consider the simplest case of a uniform resistivity, thereby removing the second term on the right hand side. Since there is no lengthscale implicit in the problem, it makes sense to recast the equation in terms of the dimensionless variable \( \varsigma \) which is defined as follows

\[
\varsigma = \frac{x}{\sqrt{4\lambda_m t}} \tag{2.35}
\]

The partial derivatives can then be rewritten

\[
\frac{\partial}{\partial x} = \frac{\partial \varsigma}{\partial x} \frac{\partial}{\partial \varsigma} = \frac{1}{\sqrt{(4\lambda_m t) \partial \varsigma}} \frac{\partial}{\partial \varsigma} \tag{2.36}
\]

\[
\frac{\partial^2}{\partial x^2} = \frac{\partial \varsigma}{\partial x} \frac{\partial}{\partial \varsigma} \left( \frac{\partial}{\partial x} \right) = \frac{1}{(4\lambda_m t) \partial \varsigma^2} \frac{\partial^2}{\partial \varsigma^2} \tag{2.37}
\]

\[
\frac{\partial}{\partial t} = \frac{\partial \varsigma}{\partial t} \frac{\partial}{\partial \varsigma} = \frac{\varsigma}{2t} \frac{\partial}{\partial \varsigma} \tag{2.38}
\]

Substituting these into equation 2.34 yields

\[
\frac{\partial \mathbf{B}}{\partial \varsigma} = \frac{\lambda_m}{2t} \frac{\partial^2 \mathbf{B}}{\partial \varsigma^2} \tag{2.39}
\]
Thus, the MFDE is recast as the following first order (in terms of $\frac{\partial B}{\partial \zeta}$) ODE

$$\frac{d^2 B}{d\zeta^2} + 2\zeta \frac{dB}{d\zeta} = 0$$

(2.40)

which has solution of form

$$\frac{dB}{d\zeta} = Ae^{-\zeta^2}$$

(2.41)

where $A$ is a constant. The magnetic field can be obtained by integrating this equation from $\zeta = 0$ (i.e. from the discontinuity in the initial magnetic field) to some arbitrary $\zeta$.

$$B = C + A \int_0^{\zeta} e^{-u^2} du$$

(2.42)

where $C$ is the constant of integration. To establish the constants $A$ and $C$ two boundary conditions are introduced. Since the diffusing magnetic field changes sign at the origin, and choosing a free, background static magnetic field to be zero, the first boundary condition can be stated: $B(x = 0, t) = 0$. Applying this constraint to eqn. 2.42 it is clear that $C = 0$. The diffusion of the magnetic field due to the discontinuity at the origin will not be felt at an infinitely distant point on the x-axis; there the field will remain unperturbed, i.e. $B(x = \pm \infty, t) = \pm B_0$. Since the integrand in eqn. 2.42 is in the form of a Gaussian function, it is natural to introduce the error function $erf$.

$$erf(\zeta) = \frac{2}{\sqrt{\pi}} \int_0^{\zeta} e^{-u^2} du$$

(2.43)

Hence it is clear, in order to meet the boundary condition as $x \to \infty$, that $A = \frac{2B_0}{\sqrt{\pi}}$ and the following expression for the magnetic field can be written

$$B(x, t) = B_0 erf(\zeta) = B_0 erf\left(\frac{x}{\sqrt{4\lambda_m t}}\right)$$

(2.44)

The $erf$ function can be expanded about the origin by employing the following MacLaurin series

$$erf(\zeta) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \zeta^{2n+1}}{n!(2n+1)}$$

(2.45)

$^4$A MacLaurin series is a Taylor series expansion of a function about 0
2.4 Diffusion of magnetic field

The magnetic diffusivity $\lambda_m$ was set to $1 \text{ m}^2\text{s}^{-1}$

the first few terms of which are

$$erf(\varsigma) = \frac{2}{\sqrt{\pi}} (\varsigma - \frac{1}{3} \varsigma^3 + \frac{1}{10} \varsigma^5 - \frac{1}{42} \varsigma^7 + \ldots)$$  (2.46)

The equation for the magnetic field evolution for $\varsigma << 1$, or equivalently $x << \sqrt{4\lambda_m t}$, may be simplified by ignoring higher order terms

$$B(x, t) \approx \frac{2B_0}{\sqrt{\pi}} \frac{x}{\sqrt{4\lambda_m t}}$$  (2.47)

The variation of the magnetic field profile as it diffuses is shown in figure 2.3. The current density profile that corresponds with this diffusing magnetic field is found by applying Ampère’s law to eqn. 2.44. Since the magnetic field has a $z$ component only and varies only along the $x$ axis, then the current density has a $y$ component only, $J = J\hat{y}$ and we have

$$J = -\frac{1}{\mu_0} \frac{dB}{dx} = -\left( \frac{B_0}{\mu_0 \sqrt{4\lambda_m t}} \right) e^{-\frac{x^2}{4\lambda_m t}}$$  (2.48)

Thus, the current density profile is homogeneous in the $y$ and $z$ direction with a Gaussian profile in the $x$-direction. Initially concentrated at the origin, the width of this Gaussian profile increases over time. The ohmic heating
2.4 Diffusion of magnetic field

Density profile, $P_\Omega = \eta J^2$, spreads out in a similar fashion (assuming uniform resistivity) - as the magnetic field (and magnetic energy density) decreases (for all $x \neq 0$), the plasma is ohmically heated.

2.4.2 Diffusion in a cylindrical geometry

In many cases it is useful to approximate the toroidal geometry of a tokamak as a long cylinder. The coordinate system, $r, \theta, z$, consists of a radial, poloidal and azimuthal coordinate. In this simple example, the magnetic geometry of the plasma is composed of concentric circular magnetic flux surfaces. The magnetic field lines lie on these flux surfaces, i.e. there is no radial component and $B = B(0, B_\theta, B_z)$. Neglecting the contribution from the convective and non-inductive current terms in eqn. 2.31, the magnetic field diffusion equation may be written

$$\frac{\partial B}{\partial t} = \frac{\eta}{\mu_0} \nabla^2 B - \nabla \left( \frac{\eta}{\mu_0} \right) \times (\nabla \times B) \quad (2.49)$$

The $z$ component of the magnetic field is, due to the assumption of an infinite aspect ratio, a constant independent of position (neglecting the small contribution to $B_z$ made by the poloidal component of the plasma current). The shape of the poloidal magnetic field $B_\theta$, on the other hand, is not as easily determined since it is dependent on the form of the unknown toroidal current density. It is necessary to recast eqn. 2.49 in cylindrical coordinates $(r, \theta, z)$ and to consider the evolution of $B_\theta$ only. To begin, consider the poloidal component of the vector Laplacian

$$\nabla^2 B_\theta = \frac{\partial^2 B_\theta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 B_\theta}{\partial \theta^2} + \frac{\partial^2 B_\theta}{\partial z^2} + \frac{1}{r} \frac{\partial B_\theta}{\partial r} + \frac{2}{r^2} \frac{\partial B_\theta}{\partial \theta} - \frac{B_\theta}{r^2} \quad (2.50)$$

Since the poloidal magnetic field is, by definition, a flux function it follows $\frac{\partial B_\theta}{\partial \theta} = 0$. Due to the assumption of translational symmetry in the $z$ direction, the derivative with respect to that coordinate is also zero. The Laplacian can be rewritten

$$\nabla^2 B_\theta = \frac{\partial^2 B_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial B_\theta}{\partial r} - \frac{B_\theta}{r^2} \quad (2.51)$$
2.4 Diffusion of magnetic field

The curl of the magnetic field in cylindrical geometry is given by

$$\nabla \times \mathbf{B} = \left( \frac{1}{r} \frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} \right) \hat{\rho} + \left( \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \right) \hat{\theta} + \frac{1}{r} \left( \frac{\partial (rB_\theta)}{\partial \theta} - \frac{\partial B_r}{\partial \theta} \right) \hat{z} \tag{2.52}$$

The resistivity $\eta$ is assumed to be a radial function $\eta = \eta(r)$ and therefore its gradient is a vector with $r$ component only. Thus, due to the cross product with the resistivity gradient in equation 2.49, only the $\theta$ and $z$ components need be considered in the the curl of the magnetic field, which by the assumptions described above further reduces to

$$\nabla \times \mathbf{B} = \frac{1}{r} \frac{\partial (rB_\theta)}{\partial r} \hat{z} \tag{2.53}$$

Since $\hat{\rho} \times \hat{z} = -\hat{\theta}$, eqn. 2.49 can be written

$$\frac{\partial B_\theta}{\partial t} = \frac{\eta}{\mu_0} \left( \frac{\partial^2 B_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial B_\theta}{\partial r} - \frac{B_\theta}{r^2} \right) + \frac{1}{r\mu_0} \frac{\partial \eta}{\partial r} \frac{\partial (rB_\theta)}{\partial r} \tag{2.54}$$

Grouping the terms together, the magnetic diffusion equation for the poloidal magnetic field in cylindrical geometry is derived

$$\frac{\partial B_\theta}{\partial t} = \frac{\eta}{\mu_0} \left( \frac{\partial^2 B_\theta}{\partial r^2} + \left[ \frac{1}{r} + \frac{1}{\eta} \frac{\partial \eta}{\partial r} \right] \frac{\partial B_\theta}{\partial r} + \left[ \frac{1}{r\eta} \frac{\partial \eta}{\partial r} - \frac{1}{r^2} \right] B_\theta \right) \tag{2.55}$$

A solution form which can be used to solve this equation is

$$B_\theta(r,t) = B_\theta^{\text{steady}}(r) + u(r)v(t) \tag{2.56}$$

where $B_\theta^{\text{steady}}(r)$ is the magnetic field profile in steady state. This is found by setting $\frac{\partial B_\theta}{\partial t} = 0$ in equation 2.55 and solving the resulting ODE. The rest of the solution is found by separating $B_\theta$ into spatial and temporal functions $u(r)$ and $v(t)$ and using the standard separation of variables technique [35] to find a solution for each function. To verify that equation 2.56 is a valid solution form for the magnetic field diffusion equation, substitute the terms into equation 2.55 to obtain

$$u \frac{dv}{dt} + \frac{dB_\theta^{\text{steady}}}{dt} = \lambda_m \left( v \frac{d^2 u}{dr^2} + \left[ \frac{1}{r} + \frac{1}{\lambda_m} \frac{\partial \lambda_m}{\partial r} \right] \frac{du}{dr} + \left[ \frac{1}{r\lambda_m} \frac{\partial \lambda_m}{\partial r} - \frac{1}{r^2} \right] uv \right)$$

$$+ \lambda_m \left( \frac{d^2 B_\theta^{\text{steady}}}{dr^2} + \left[ \frac{1}{r} + \frac{1}{\lambda_m} \frac{\partial \lambda_m}{\partial r} \right] \frac{dB_\theta^{\text{steady}}}{dr} + \left[ \frac{1}{r\lambda_m} \frac{\partial \lambda_m}{\partial r} - \frac{1}{r^2} \right] B_\theta^{\text{steady}} \right)$$
Since by definition $\frac{d}{dt}B^\text{steady}_\theta = 0$ and the terms involving $B^\text{steady}_\theta$ on the right hand side add up to zero, one may write

\[
\frac{1}{v} \frac{dv}{dt} = \lambda_m \left( \frac{1}{u} \frac{d^2 u}{dr^2} + \left[ \frac{1}{r} + \frac{1}{\lambda_m} \frac{\partial \lambda_m}{\partial r} \right] \frac{1}{u} \frac{du}{dr} + \left[ \frac{1}{r} \frac{\partial \lambda_m}{\partial r} - \frac{1}{r^2} \right] \right)
\]  
(2.57)

which is the equation that results if $B_\theta(r,t) = u(r)v(t)$ is substituted into equation 2.55, i.e. it is valid to add a $B^\text{steady}_\theta$ profile to the solution.

**Uniform diffusivity**

In this section a solution to equation 2.55 will be found in the case of uniform diffusivity $\lambda_m(r) = \lambda_o$. The magnetic field diffusion equation becomes

\[
\frac{\partial B_\theta}{\partial t} = \lambda_o \left( \frac{\partial^2 B_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial B_\theta}{\partial r} - \frac{1}{r^2} B_\theta \right)
\]  
(2.58)

The solution form given by equation 2.56 will be used. In order to proceed, the solution to the magnetic field diffusion equation in steady state will first be found and then the product of the separated variables added.

In order to solve equation 2.58 some initial and boundary conditions need to be prescribed.

\[
B_\theta = B^\text{bnd}_\theta, \quad r = a, \quad t \geq 0,
\]  
(2.59)

\[
B_\theta = 0, \quad r = 0, \quad t \geq 0,
\]  
(2.60)

\[
B_\theta = f(r), \quad 0 < r < a, \quad t = 0
\]  
(2.61)

where $a$ is the radius of the last closed flux surface of the plasma and $f(r)$ is a function describing the initial $B_\theta$ profile.

In steady state ($\frac{\partial B_\theta}{\partial t} = 0$) and assuming uniform diffusivity ($\frac{\partial \lambda_m}{\partial t} = 0$), equation 2.58 becomes

\[
\frac{\partial^2 B_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial B_\theta}{\partial r} - \frac{1}{r^2} B_\theta = 0
\]  
(2.62)

The only solution to this equation which is consistent with the boundary conditions is

\[
B^\text{steady}_\theta = B^\text{bnd}_\theta \left( \frac{r}{a} \right)
\]  
(2.63)

Next, find solutions for the separated variables. By substitution of equation 2.56 into the diffusion equation, dividing by $uv$ and setting both sides
equal to the constant $\alpha^2 \lambda_o$, one obtains ordinary differential equations in terms of $u$ and $v$.

$$\frac{1}{v} \frac{\partial v}{\partial t} = -\alpha^2 \lambda_o$$  \hspace{1cm} (2.64)

$$\lambda_o \left( \frac{1}{u} \frac{\partial^2 u}{\partial r^2} + \frac{1}{u r} \frac{\partial u}{\partial r} - \frac{1}{r^2} \right) = -\alpha^2 \lambda_o$$  \hspace{1cm} (2.65)

The solution to equation 2.64 is given by

$$v = e^{-\alpha^2 \lambda_o t}$$  \hspace{1cm} (2.66)

Equation 2.65 may be written in the following form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \left( -\frac{1}{r^2} + \alpha^2 \right) u = 0$$  \hspace{1cm} (2.67)

This is Bessel’s equation of the first kind, of the first order. In general, its solution is a linear combination of Bessel functions of the first and second kind (the respective symbols of these functions being $J_\nu(\alpha r)$ and $Y_\nu(\alpha r)$).

$$u(r) = C_1 J_1(\alpha r) + C_2 Y_1(\alpha r)$$  \hspace{1cm} (2.68)

where $C_1$ and $C_2$ are constants of integration and $\nu$ denotes the order of the Bessel function.

For small values of $r$, the Bessel functions of the first and second kind may be approximated as follows $J_1(\alpha r) \approx \frac{\alpha r}{2}$ and $Y_1(\alpha r) \approx -\frac{2}{\pi \alpha r}$. To avoid the unphysical singularity at the magnetic axis in $Y_1(\alpha r)$, the constant $C_2$ is set to zero. Thus, the solution to the diffusion equation will have the following form

$$B_\theta(r, t) = B_\theta^{\text{bnd}} \left( \frac{r}{a} \right) + \sum_{n=1}^{\infty} A_n J_1(\alpha_n r) e^{-\lambda_o \alpha_n^2 t}$$  \hspace{1cm} (2.69)

where the series $\alpha_n a$ are the tabulated roots of $J_1(z)$. This equation satisfies the boundary condition of a constant $B_\theta$ at the edge and zero $B_\theta$ at the magnetic axis. It also satisfies the criterion that the steady state profile is reached as $t \rightarrow \infty$. In order to satisfy the initial condition (eqn. 2.61) it is apparent that

$$f(r) = B_\theta^{\text{bnd}} \left( \frac{r}{a} \right) + \sum_{n=1}^{\infty} A_n J_1(\alpha_n r)$$  \hspace{1cm} (2.70)

is required. The $A_n$ coefficients are determined by multiplying both sides of eqn. 2.70 by $r J_1(\alpha_n r)$ and integrating from 0 to $a$. Rearranging in terms of
2.4 Diffusion of magnetic field

the coefficients one has

$$
\sum_{n=1}^{\infty} A_n = \frac{a \int_0^a r f(r) J_1(\alpha_n r) dr - B_{\theta}^{\text{bnd}} \int_0^a r^2 J_1(\alpha_n r) dr}{\int_0^a r J_1(\alpha_n r) J_1(\alpha_m r) r dr} (2.71)
$$

Bessel functions are orthogonal in the interval \([0, a]\) according to

$$
\int_0^a J_\nu(\alpha_m r) J_\nu(\alpha_n r) r dr = \frac{1}{2} a^2 [J_{\nu+1}(\alpha_m a)]^2 \delta_{mn} (2.72)
$$

where \(\delta_{mn}\) is the Kronecker delta \([34]\). In addition, it can be shown

$$
\int_0^a r^2 J_1(\alpha_n r) dr = \frac{a^2 J_2(\alpha_n a)}{\alpha_n} (2.73)
$$

Therefore, eqn. 2.71 may be rewritten

$$
\sum_{n=1}^{\infty} A_n = \frac{2 \int_0^a f(r) J_1(\alpha_n r) dr}{a^2 [J_2(\alpha_n a)]^2} - \frac{2 B_{\theta}^{\text{bnd}} J_2(\alpha_n a)}{\alpha_n a [J_2(\alpha_n a)]^2} (2.74)
$$

By substituting this expression into eqn. 2.69, the solution to the magnetic field diffusion equation (eqn. 2.58) in the case of uniform diffusivity and consistent with the boundary and initial conditions given by eqn. 2.59 to eqn. 2.61, is found

$$
B_\theta(r,t) = \frac{r B_{\theta}^{\text{bnd}}}{a} + \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{J_1(\alpha_n r) e^{-\lambda_n \alpha_n^2 t}}{[J_2(\alpha_n a)]^2} \left( \int_0^a f(r) J_1(\alpha_n r) dr - \frac{a B_{\theta}^{\text{bnd}} J_2(\alpha_n a)}{\alpha_n} \right) (2.75)
$$

If, apart from at the boundary, \(B_\theta\) is initially zero throughout the cylinder then \(f(r) = 0\) and the magnetic field is given by

$$
\frac{B_\theta(r,t)}{B_{\theta}^{\text{bnd}}} = \frac{r}{a} - \frac{2}{a} \sum_{n=1}^{\infty} \frac{J_1(\alpha_n r) e^{-\lambda_n \alpha_n^2 t}}{\alpha_n J_2(\alpha_n a)} (2.76)
$$

The current density profile can be found using Ampere’s Law which in this case reduces to

$$
J_z = \frac{1}{r \mu_0} \frac{\partial (r B_\theta)}{\partial r} (2.77)
$$

Since we have \(\frac{\partial}{\partial r} J_1(\alpha_n r) = \frac{\alpha_n}{2} (J_0(\alpha_n r) + J_2(\alpha_n r))\), it follows

$$
J_z = \frac{2 B_{\theta}^{\text{bnd}}}{\mu_0 a} \left( 1 - \sum_{n=1}^{\infty} \left( \frac{J_0(\alpha_n r)}{2} + \frac{J_1(\alpha_n r)}{\alpha_n r} \right) \frac{1}{J_2(\alpha_n a)} e^{-\lambda_n \alpha_n^2 t} \right) (2.78)
$$
Note that at steady state \((t \to \infty)\), the second term on the right hand side tends to zero and the current density is constant across the plasma
\[ J_{z}^{\text{steady}} = \frac{2B_{0}^{\text{ind}}}{\mu_{0}a}. \]
The magnetic field diffusion equation is solved for cases of non-uniform diffusivity in section 4.7.

2.4.3 Plasma electrical resistivity

In order to determine the magnetic diffusivity \(\lambda_{m} = \frac{\eta}{\mu_{0}}\), the electrical resistivity of the plasma must be found. At the drift velocity, \(v_{d}\), the force on the electrons in a plasma due to an applied electric is balanced by the retarding collisional force.

\[ Ee = \frac{m_{e}v_{d}}{\tau_{c}} \]  
(2.79)

where \(m_{e}\) is the electron mass and \(\tau_{c}\) is the electron-ion collision time. Since the resistivity is defined via Ohm’s Law as \(\eta = \frac{E}{J}\) and since we have \(J = nev\), we may write [9]

\[ \eta = \frac{m_{e}}{n_{e}e^{2}\tau_{c}} \]  
(2.80)

where \(n_{e}\) is the electron density. From the Fokker-Planck equation it can be shown that the characteristic collision times in a plasma have the following form

\[ \tau_{c} \propto \frac{\varepsilon_{0}^{2}m^{1/2}T^{3/2}}{ne^{4}ln\Lambda} \]  
(2.81)

Spitzer found that the effective collision time of the electrons, when one takes electron–electron collisions into account, is roughly half the electron-ion collision time \(\tau_{e}\). Spitzer’s formula [36] for the electrical resistivity in plasma appears below. Where there is an impurity population, \(Z_{\text{effective}}\) should be used in place of the atomic number \(Z\). \(N(Z)\) is a prescribed function of \(Z\).

\[ \eta_{\text{Spitzer}} = 0.51\frac{ZN(Z)m^{1/2}e^{2}ln\Lambda}{3\varepsilon_{0}^{2}/(2\pi T^{3/2})} \]  
(2.82)

where \(ln\Lambda\), the Coulomb logarithm, varies weakly with plasma density and temperature. The Spitzer resistivity is valid in a plasma where there is no magnetic field or when considering current flowing parallel to the magnetic field. Due to the tendency of the electrons to gyrate about the magnetic field lines (see section 2.1), currents flowing perpendicular to the magnetic field are subject to a resistivity almost double the Spitzer value.
2.5 Neoclassical theory

2.5.1 Transport

The theory of transport deals with the motion of particles and energy across the plasma magnetic field lines. An unavoidable component of such transport is that due to Coulomb collisions. In a cylindrically symmetric plasma geometry this diffusion is known as classical transport. In the reduced symmetry of a toroidal magnetic geometry, the increase in the rate of transport is known as neoclassical [9].

In order to study particle transport consider that the equation of the continuity (eqn. 2.6) may be written as follows

\[
\frac{\partial n_i}{\partial t} + \nabla \cdot \Gamma_i = S_i
\]

(2.83)

where \( \Gamma_i \) is the particle flux and \( S_i \) is the particle source or sink term for plasma species \( i \). The particle flux is given by

\[
\Gamma_i = n_i V_{\text{conv}} - D \nabla n_i
\]

(2.84)

where the first term on the R.H.S is the convective term (\( V_{\text{conv}} \) is the convective velocity), and the second term is due to particle diffusion (\( D \) being the particle diffusivity) [37]. The diffusivity may be described by considering the motion of the particle as a random walk. Note that

\[
D_{\text{random walk}} = (\text{step length})^2 \times (\text{collision frequency})
\]

(2.85)

In the case of classical transport the step length in the radial direction is given by the radius of the electron gyration about the magnetic field line, i.e. the electron Larmor radius \( \rho_e \). Therefore

\[
D_{\text{classical}} = \frac{\rho_e^2}{\tau_e}
\]

(2.86)

is the classical diffusivity term where \( \tau_e \) is the electron collision time given by equation 2.81. In the neoclassical case the existence of banana orbits (described in the next section) means that the maximum possible radial excursion executed by a particle is increased. The neoclassical diffusivity for a plasma in which a high proportion of particles are in banana orbits is given
by

$$D_{\text{neo}} \approx \Delta r_b^2 \nu_c$$  \hfill (2.87)

where $\Delta r_b$ is the radial width of the banana orbit and $\nu_c$ is the collision frequency. Since $\Delta r_b$ is typically much greater than the Larmor radius, the presence of trapped particles results in an enhanced diffusivity over the classical case for a given collision frequency. The effect of turbulence in the electric and magnetic fields in the plasma leads to even greater transport than that predicted by neoclassical theory; so called anomalous transport. This effect has been observed in many plasma devices, including JET [38].

2.5.2 Particle trapping in a tokamak

\[ \text{Figure 2.4: The gradient in the toroidal magnetic field seen by a charged particle. Source [1].} \]

In a tokamak the toroidal magnetic field increases with decreasing major radius (see figure 2.4). It can be shown that as a charged particle moves in a slowly-varying magnetic field, its magnetic moment is an adiabatic constant of the motion

$$\mu_m = \frac{mv_{\perp}^2}{2B}$$  \hfill (2.88)

where $v_{\perp}$ is the velocity of the particle perpendicular to $B$, i.e. the orbital velocity of the particle as it gyrates about the magnetic field line. If, in
addition, the plasma is sufficiently collisionless that the charged particle can move from the inner to the outer edge of the flux surface to which it is tied (i.e. the distance known as the connection length) without an impact, then the kinetic energy of the particle is a constant of the motion over this distance. Thus, as the particle moves into the high-field side of the plasma its perpendicular velocity increases at the expense of its parallel velocity. If the parallel velocity reduces to zero, then the direction of the particle’s motion along the magnetic field line reverses, i.e. the particle is reflected and returns to the low field side of the plasma. Due to the up-down symmetry of the tokamak magnetic field, the particle is again reflected at the opposite extremum of its trajectory, completing a so-called ‘banana orbit’. Such a particle is said to be ‘trapped’ in the low-field side of the tokamak plasma (as opposed to a ‘passing’ particle, where the ions moves freely around the major circumference of the tokamak on nested drift surfaces slightly shifted with respect to the magnetic surface. The trajectory of a passing particle, projected on the poloidal plane, is sometimes described as an ‘orange’ orbit).

A low-collisional plasma, and hence one in which a substantial number of particles are in trapped orbits, is said to be in the banana regime. A highly-collisional plasma, on the other hand, is in what is known as the Pfirsch-Schlüter regime. Particles in this regime on average experience a collision before completing a connection length. Between the banana and Pfirsch-Schlüter regimes (in terms of collision frequency) lies the plateau regime.\footnote{The particle diffusion coefficient rises with collisionality in the banana and Pfirsch-Schlüter regimes but is independent of collisionality in the plateau regime}

Whether a charged particle is in a trapped or passing trajectory depends on the pitch of its velocity vector, $\frac{|v_\parallel|}{|v_\perp|}$, with respect to the local magnetic field. In the simplest case of circular flux surfaces, the velocity pitch of a trapped particle at the midplane will satisfy the following condition\ [9]

$$\frac{v_\parallel}{v_\perp} < \sqrt{\frac{2r}{R_0}}$$

(2.89)

where $r$ and $R_0$ are the minor and major radii of the flux surface respectively at the magnetic midplane. Since this fraction of trapped particles are restricted in their motion along the magnetic field line, the overall current is reduced. The trapped particle fraction diminishes with increasing collisionality.
2.5 Neoclassical theory

Poloidal cross-section

Major radius

Figure 2.5: Banana orbit executed by a trapped particle. Source [1]

The trapped particle effect in its description of tokamak plasmas (including various formulations of the resistivity).

2.5.3 Bootstrap current

Neoclassical theory predicts a radially inward particle flux of trapped particles known as the Ware pinch effect [39]. The Onsager reciprocal relations require the existence of a parallel electric current driven by a density gradient [9]. This diffusion-driven current is known as the bootstrap current and was predicted by theorists [40], [41] before its existence was confirmed by experimental observation (for example at JET [42]). The bootstrap moniker alludes to the fact that the current is neither inductive nor dependent on any external current drive, instead being driven by pressure gradients in the plasma. The bootstrap current typically comprises a significant proportion of total plasma current in steady state operation. For example, at JT-60, fully non-inductive current drive was achieved with the bootstrap current contributing 70 – 80% of the total plasma current and neutral beam current drive making up the remainder [14]. Total bootstrap current of 1MA has been attained at JET, representing 50% of the total plasma current [43].

6In thermodynamics, the Onsager reciprocal relations express the equality of certain relations between flows and forces in thermodynamical systems out of equilibrium, but where a notion of local equilibrium exists. The fluxes $\Gamma_i$ are related to thermodynamic forces $F_j$ via the equation $\Gamma_i = \sum_j L_{ij} F_j$ where $L_{ij}$ is a symmetric matrix of coefficients.
2.5.4 Neoclassical models of resistivity and bootstrap current

There are a number of formulations of neoclassical resistivity and bootstrap current. Early formulations are valid over certain ranges of aspect ratio and collisionality (beginning with the work of Hinton and Hazeltine [44]). For example, the Hirshman-Hawryluk-Birse [45] model for the resistivity and the Hirshman formulation of the bootstrap current [46] are valid for arbitrary inverse aspect ratios in the banana regime but only for low inverse aspect ratios in the other collisional regimes.

The NCLASS model [47] follows the multi-species fluid, reduced charge state approach originally adopted by Hirshman and Sigmar [48]. Using this method, NCLASS solves the radial and parallel force balance equations for the flows and neoclassical viscosities within a flux surface. From these flows, the bootstrap current and parallel electrical resistivity are derived, being given by

$$\eta_{\text{neo}} = \frac{<E \cdot B>}{\sum a_i e_{a_i} n_{a_i} \hat{r}_{E}^{a_i}}$$

$$<J_{BS} \cdot B> = \sum a_i e_{a_i} n_{a_i} \hat{u}_{pT}^{a_i}$$

where $e_{a_i}, n_{a_i}$ represent the charge and density of species $a$ at charge state $i$. $\hat{r}_{E}^{a_i}$ is the source term due to the inductive electric field while $\hat{u}_{pT}^{a_i}$ is the flow due to pressure and temperature gradients. Since the viscosities are calculated by numerically integrating over velocity space, NCLASS is valid over all collisionality regimes and aspect ratios (an improvement on the Hirshman and Sigmar model).

Although the multi-species NCLASS model of the neoclassical resistivity and bootstrap current is considered the most comprehensive, its comparison with the simpler Zeff-parameterized Sauter model [49] (see also the corrigendum [50]) is of interest. The Sauter model is as general as NCLASS in its range of applicability, not being limited to any range of aspect ratio or collisionality. The Sauter resistivity is related to the Spitzer value via the parameter $F_{33}$ in [49]

$$F_{33} = 1 - (1 + \frac{0.36}{Z})X + \frac{0.59}{Z}X^2 - \frac{0.23}{Z}X^3$$

where $Z$ is the ion charge and $X$ is a parameter related to the trapped particle
fraction. The Sauter resistivity is given by

\[ \eta_{\text{neo}} = \frac{\eta_{\text{Spitzer}}}{F_{33}} \]  

(2.93)

The Sauter bootstrap current can be written as follows

\[ \langle J \parallel B \rangle = \sigma_{\text{neo}} \langle E \parallel B \rangle - I(\psi) p_e \left[ L_{31} \frac{p}{p_e} \frac{\delta \ln p}{\delta \psi} + L_{32} \frac{\delta \ln T_e}{\delta \psi} + L_{34} \alpha \left( \frac{1 - R_{\text{pe}}}{R_{\text{pe}}} \right) \frac{\delta \ln T_i}{\delta \psi} \right] \]  

(2.94)

where \( I(\psi) = R B \phi \), \( p \), \( p_e \) is the total pressure and electron pressure respectively and \( L_{31}, L_{32}, L_{34} \) and \( \alpha \) are coefficients depending on the trapped fraction \( f_t \) and on the electron and ion collisionalities; \( v_e \) and \( v_i \).

It is evident from equation 2.89 that the fraction of trapped particles in a tokamak plasma is inversely related to the flux surface aspect ratio \( \left( \frac{B}{r} \right) \). For instance, the last closed flux surface at JET (i.e. the plasma boundary) typically has an aspect ratio \( \approx 3 \) which implies that the condition for trapping is \( \frac{v_i}{v_e} < 0.8 \). Consequently, the majority of particles in the outer regions of a JET plasma are trapped, leading to substantial and measurable neoclassical effects. Indeed, results from JET and several other tokamaks indicate the existence of the bootstrap current and that the resistivity is consistent with neoclassical theory.
Chapter 3

EFIT equilibrium code

3.1 Introduction

EFIT (Equilibrium Fitting) is a free-boundary MHD equilibrium reconstruction code. It was developed by Lang Lao for use on the DIIID tokamak and, with modifications to suit each tokamak, is now in widespread use. The code is a successor of MFIT (Magnetic Fitting), the essential innovation being in the interleaving of the equilibrium and fitting iterations and in allowing a distributed plasma current source [51]. With this approach, an equilibrium reconstruction requires only the equivalent of 1-2 equilibrium iteration cycles, substantially reducing the amount of time required to find the optimum solution compared with conventional equilibrium codes. The code is run both automatically as part of the intershot suite of codes at JET, and off-line for a more in-depth analysis of the equilibrium. A real-time version of the code is under development. As well as utilising the magnetic measurements external to the plasma, EFIT is capable of including internal motional Stark effect, Faraday rotation and kinetic pressure measurements in its determination of the equilibrium. The Grad-Shafranov equilibrium equation, which describes the force balance in a plasma, is solved using the available measurements as constraints on the toroidal current density profile. Quantities output by EFIT include the plasma stored energy, the magnetic flux surface geometry and the plasma current density profile.
3.2 Equilibrium solution methods

In order to solve the Grad-Shafranov equation (equation 2.26), an elliptic partial differential equation, some boundary conditions need to be specified. This may involve prescribing the flux, or normal derivatives of the flux, on some boundary in the poloidal plane which surrounds the plasma (generally the location of this bounding contour will be close to the set of magnetic flux and local magnetic field measurements). Alternatively, the boundary conditions may specify that the poloidal flux go to zero on axis and at infinity. In free-boundary equilibrium codes such as EFIT and CLISTE [52], the plasma boundary is found as part of the equilibrium solution. The plasma boundary is either the largest closed contour of $\psi$ to touch the limiter or can be identified by the appearance of a null point in the case of a divertor geometry.

The solution to the Grad-Shafranov is an incompletely posed one in that the $p'$ and $FF'$ basis functions cannot be uniquely determined without further restrictions on those profiles [53]. However, if the external magnetic measurements provide sufficient redundancy, beyond that required to solve the Grad-Shafranov equation in the case where the current profile is known, a solution for the basis functions $p'$ and $FF'$ may be found [54]. Hence the gross features of the toroidal current density profile $j_\phi$, particularly towards the plasma boundary, may be elucidated. In all MHD equilibrium codes, some parameterisation of the basis functions $p'$ and $FF'$ in $\psi$ is chosen. This parameterisation may take the form of a polynomial, a power function, an exponential or some spline representation. In EFIT the former and latter representations are employed.

On solving the Grad-Shafranov equation, and thereby calculating a current profile, the field or flux at the measurement locations may be computed using Green’s functions. The discrepancy that results summed up over all the measurements (and fitting constraints) is encapsulated in the figure of merit quantity $\chi^2$ (also known as the cost function) where

$$\chi^2 = \sum_{i=1}^{N_m} \frac{M_i - \hat{M}_i}{\sigma_i^2} + \sum_{i=1}^{N_c} \frac{C_i - \hat{C}_i}{\xi_i^2}$$

(3.1)

where $N_m$ and $N_c$ represent the total number of measurements and constraints (the 'constraints' in this case include all non-measurement restric-
tions on the current profile, e.g. curvature penalties) involved in the reconstruction. $M_i$ and $C_i$ denote the i-th measurement and constraint respectively while $\hat{M}_i$, $\hat{C}_i$ denote the corresponding quantities as predicted by the equilibrium code. Finally, $\sigma_i$ and $\xi_i$ denote the inverse weights associated with the i-th measurement and constraint. The equilibrium code, upon each iteration, will adjust parameters in the current profile representation in order that the cost function be minimized while still satisfying the Grad-Shafranov equation.

### 3.3 EFIT solution technique

In the case of EFIT, the Grad-Shafranov equation is solved over an infinite domain. All sources of current, including those in the external coils and the iron core, are included in the cost function (eqn. 3.1). The equilibrium equation is solved by way of a Picard iteration scheme first proposed in [55]. This involves transforming the original non-linear equation (eqn. 2.26) into a series of linearized minimisations interleaved with the equilibrium iterations [27]. The source terms for the plasma toroidal current density profile $J_\psi$, namely $p'(\psi)$ and $FF'(\psi)$, are generally parameterised in terms of a polynomial function of normalised flux $\hat{\psi}$

\[
p'(\psi) = \sum_{k=1}^{N_p} C^P_k \hat{\psi}^{k-1} - \alpha \hat{\psi}^{N_p} \sum_{k=1}^{N_p} C^P_k
\]

\[
FF'(\psi) = \sum_{k=1}^{N_F} C^F_k \hat{\psi}^{k-1} - \alpha \hat{\psi}^{N_F} \sum_{k=1}^{N_F} C^F_k
\]

where $\hat{\psi} = (\psi - \psi_{ax})/(\psi_{bd} - \psi_{ax})$ and $C^P$, $C^F$ are free parameters to be determined. The order of the polynomials $N_p$ and $N_F$ are selected by the user. The switch $\alpha$ may be set either to zero or unity; the former allows a finite current at the plasma edge. The first step in the EFIT iteration scheme is, for a given flux configuration $\psi^m$ (the flux function at iteration $m$), to determine those free parameters which characterise the plasma current $C^P$, $C^F$ that minimise the cost function $\chi^2$ (eqn. 3.1). This is done using singular value decomposition. Simultaneously, the free parameters corresponding to currents in the external coils and iron core are found. The inverse weights associated with the measurements $\sigma_i$, ensure that the fit to the measure-
ments reflects their experimental uncertainty. Thus, a toroidal current density $J_\phi(R, \psi^m)$ may be calculated, via the basis functions (eqns. 3.2 and 3.3), as a linear function of $\psi^m$.

The next step is to compute the poloidal flux function at iteration $m + 1$ due to the current density profile found at iteration $m$

$$\Delta^* \psi^{m+1} = -\mu_0 R J_\phi(R, \psi^m)$$

(3.4)

![Figure 3.1: EFIT calculated current density profile and cost function $\chi^2$ after 1–5 iterations. In order of increasing number of iterations the current density profile is represented by the following lines - bold, dashes, dots, dash-dot and dash-dot-dot. Shot 51675.](image)

The fast Buneman’s [56] method is then used to find the updated poloidal flux values at the interior grid points (a rectangular grid of size $33 \times 33$ or $65 \times 65$ is employed at JET). Having found a new poloidal flux map the current density profile may be updated as already described to produce $J_\phi(R, \psi^{m+1})$. This process will continue for a prescribed maximum number of iterations or until the change in the solution $\psi$ between successive iterations drops below some user-defined threshold (see figure 3.1).

### 3.4 Input Data

A minimal EFIT run utilises data from the magnetic field and flux sensors, measurements of the total plasma current and any external currents (including simulated skin currents in the iron core (section 3.5.1)). The addition
of Faraday rotation and motional Stark effect data (MSE) from polarimetry measurements enable the shape of the plasma current profile to be found with increased accuracy. A weight is ascribed to each measurement, denoting the importance that the code will attach to it when carrying out each minimization of the cost function. Setting a particular weight close to zero, ensuring that the data is input but then ignored when arriving at an equilibrium solution, allows a prediction of that quantity by the code.

The parameterization of the current density profile (or to be more specific, of the source functions $p'$ and $ff'$), and the curvature penalty imposed, is tailored to the data included in a particular EFIT run. When including magnetic sensor data only, a low order polynomial will suffice to model the current density profile. For example, in figure 3.2 the source functions for the magnetics only case were parameterized by setting $N_P = N_F = 2$ in equations 3.2 and 3.3 respectively and setting $\alpha$ to unity in both equations (i.e. forcing edge current to zero). In order to fit to the MSE data, a parameterization allowing a higher degree of freedom is generally required. In the figure above, this was achieved by employing a spline representation. Two and five internal spline knots respectively were used to model the $p'$ and

Figure 3.2: Current profile density calculated by EFIT when constrained by (i) magnetics data only (bold), (ii) MSE data in addition (dashes), and (iii) with Faraday rotation data in addition (dashes+dots). Shot number 53492 at 45.5s.
3.4 Input Data

Figure 3.3: Magnetic shear profile calculated by EFIT when constrained by (i) magnetics data only (bold), (ii) MSE data in addition (dashes), and (iii) with Faraday rotation data in addition (dashes+dots). Shot number 53492 at 45.5s.

The dip in the current profile around the magnetic axis is elucidated by appropriate positioning of the knots. As is clear from figures 3.2 and 3.3, a reversed shear\(^1\) current profile is maintained even with the addition of Faraday rotation data.

### 3.4.1 Magnetic Sensors

The array of magnetic sensors (collectively known as KC1D) regularly used in the EFIT calculation, consists of 39 pick-up coils (see figure 3.4), 2 full flux loops and 28 saddle loops (see figure 3.5). 18 of the pick-up coils are fixed to the in-vessel wall of each octant and measure the poloidal magnetic field at points circumscribing the plasma. Each coil consists of a coaxial cable with copper core wound hundreds of times and contained in an Inconel 625 (a heat resistant alloy consisting mainly of nickel and chromium) tube [57]. Fourteen further pick-up coils located around the divertor, oriented in the poloidal or radial directions, are necessary to resolve the geometry of the field lines in

\(^1\)The magnetic shear \(s\) is a measure of the rate of change of the magnetic field lines between neighbouring flux surfaces and is given by \(s = \frac{\partial q}{\partial \rho} \frac{\partial q}{\partial \rho}\) where \(\rho\) is the normalised radius.
that region. Five poloidal limiter coils and two limiter coils that are regularly used in EFIT have a fast frequency response (up to 500 kHz) enabling them to also be used in real-time control of the plasma magnetic equilibrium and in the study of MHD activity. A typical EFIT run will utilise 150 time points from the magnetics data.

Two full flux loops remain from an original seven. They are silicon insulated wires, enclosed in tubes, that complete a toroidal circuit of the vessel. Fourteen saddle loops are attached to the external wall of each octant of the vacuum vessel. They are arranged in order to measure the magnetic flux through virtually the entire surface area of the vessel. A further twenty four saddle loops measure the flux in the divertor region. New magnetic sensors will be installed during the current JET shutdown (2004-5). In particular, Hall probes will be added in order to measure the residual magnetization of the iron limb.

### 3.4.2 Motional Stark effect polarimetry

A Lorentz field is induced on an atom moving across a magnetic field causing spectral line splitting and linear polarisation of the emitted radiation. The
3.4 Input Data

JET MSE diagnostic (known as KS9) measures the polarisation of the Stark-split $D_\alpha$ emission from deuterium atoms injected by neutral heating beams, and can provide accurate measurements of local magnetic pitch angle $\gamma$ [58]. Pitch angle measurements in the plasma allow fine structure in the current profile to be resolved. The JET MSE diagnostic has 25 channels in the interval $2.68 \text{m} < R < 3.88 \text{m}$ (where R is the major radius). Due to the fact that the KS9 viewing optics and the neutral beams are not horizontal and not in the same plane as the magnetic axis, the angle measured $\gamma_m$ is a complicated expression of all the magnetic field components parameterised as follows [59]

$$\tan \gamma_m = \frac{E_H}{E_V} = \frac{B_V A_0 + B_R A_1 + B_T A_2}{B_V A_3 + B_R A_4 + B_T A_5}$$ \hspace{1cm} (3.5)

where $E_H$ and $E_V$ are the horizontal and vertical electric field components in the coordinate frame of the polarimeter line of sight, $B_V, B_R, B_T$ are the magnetic field components and the coefficients $A_0 - A_5$ are geometric factors. Thus, EFIT can derive the magnetic pitch angle $\gamma$ from the measured angle $\gamma_m$ using equation 3.5. The design of the neutral beam injection system...
3.4 Input Data

complicates the operation of the diagnostic [60]. If an unfavourable combination of PINIs are fired simultaneously, the interpretation of the KS9 signal can be problematic. Some outboard channels are ignored due to the limitations of the viewing geometry of the MSE diagnostic. In addition, some of the inboard channels may suffer from a low signal-to-noise ratio due to the attenuation of the neutral beams. The radial electric field can affect the interpretation of the MSE measurements [61]. This correction is significant in plasmas with large rotation velocities or pressure gradients. A technique has been developed to measure $E_r$ and thus improve the MSE pitch angle measurements [62].

3.4.3 Faraday rotation polarimetry

The multichannel polarimetry diagnostic at JET shares its 4 vertical and 4 lateral lines of sight with the Far Infra-Red (FIR) interferometer and measures the line-integrated product of the electron density $n_e$ and the parallel component of the poloidal magnetic field $B_\parallel$. Neglecting the Cotton-Mouton effect (which is not negligible at JET [63]), the Faraday rotation angle is
given by
\[ \Delta \Gamma = C \lambda^2 \int l_n l B_{\parallel}(l) dl \] (3.6)
where \( C \) is a constant \( (C = 2.615 \times 10^{-13} T^{-1}) \) and \( \lambda \) is the wavelength of the laser beam. Since EFIT is also supplied with the interferometric density measurements along the same lines of sight, the code can attempt to fit these internal measurements of the magnetic field. The error (one standard deviation) associated with the measured Faraday rotation angle is normally 0.2°. The polarimetry lines of sight, projected onto the poloidal plane, are shown in figure 3.6.

3.5 EFIT Iron model

3.5.1 Iron model description

EFIT calculates the magnetic equilibrium over a full domain. Thus, it is necessary to include all currents; including those induced in the ferromagnetic iron core. Since there are no measurements of these currents, they must be modelled in order that their influence on the magnetic flux geometry be taken into account by EFIT. The iron model in the code is included in the subroutine DOIRON.

In DOIRON the iron is treated as 48 separate thin segments, each with a uniform surface current. In the model, about 1 cm from each iron segment there are simulated magnetic probes, 48 tangential and 48 normal to the iron-air interface. This is to facilitate the imposition of the boundary condition that the normal component of the poloidal flux \( \psi \) is continuous across the interface, while the tangential component of the magnetic flux density \( B_t \) is discontinuous if the relative permeability of the iron is greater than unity. Thus, at each surface element \( \lambda_i \) we have the following boundary conditions

\[ \psi^{\text{iron}}(\lambda_i) = \psi^{\text{air}}(\lambda_i) \] (3.7)
\[ B_{t}^{\text{iron}}(\lambda_i) = \mu_r(B)B_{t}^{\text{air}}(\lambda_i) \] (3.8)

Combining these two equations, an expression for the iron surface current at
each segment may be derived [64]

\[
[\mu_r(B_i) - 1][B_i^{\text{ext}}(\lambda_i) + \sum_j G_B(\lambda_i, \lambda_j)I_{mj}] - [\mu_r(B_i) + 1] \left[ \frac{\mu_0 I_{mi}}{2L_i} \right] = 0 \tag{3.9}
\]

where \( B_i^{\text{ext}}(\lambda_i) \) is the tangential poloidal magnetic field due to the plasma, shaping coils and ohmic coils and \( I_{mi} \) is the total surface magnetization current in segment \( \lambda_i \). The permeability is calculated as a prescribed function of the magnetic field in the code. \( G_B(\lambda_i, \lambda_j) \) is the inductance matrix relating the tangential poloidal magnetic field at segment \( \lambda_i \) to the magnetization current \( I_{mj} \) flowing in segment \( \lambda_j \) and \( L_i \) is the length of segment \( \lambda_i \).

The algorithm followed by the code is as follows

(a) Calculate the tangential and normal magnetic fields at the simulated probes due to the external currents. The magnitude of the total magnetic field (i.e. \( B_i \) in equation 3.9) is then found by summing these two vectors.

(b) Calculate the relative permeability \( \mu_r(B_i) \) at each segment \( i \) using the function described below (section 3.5.2).

(c) Find the iron magnetisation currents using LU decomposition.

(d) Recalculate the magnetic fields at the probes, including the effect of the iron currents (using a pre-calculated set of Greens functions).

(e) Repeat steps (b) to (d) until the iron currents converge after a few iterations.

Thus, a set of iron magnetisation currents consistent with the external currents and magnetic measurements is obtained. It should be noted that the remanent magnetisation, i.e. the magnetization left behind in the iron after the external magnetic field is removed, is not measured and not included in the model.

The geometry of the iron in the model is based on the equivalent 2-D axisymmetric representation of the actual 3-D structure [65]. The 2-D model is based on detailed comparisons of simulated flux values with a 3D code. One constraint on the model is that the total volume of transformer iron should be equal in both calculations. In the model the upper and lower sections are gently inclined when in fact they are horizontal. This modification take into account the relative decrease in the amount of iron per radian with increasing distance from the tokamak center. Furthermore, the outer limb of the iron as modelled is more distant from the plasma than it is in reality.
3.5 EFIT Iron model

Figure 3.7: Poloidal cross-section showing real iron core (dotted line), 2-D representation used in EFIT iron model (bold), centre of EFIT iron current segments (squares), shaping coils and vacuum vessel (see figure 3.7). This change was made to reduce the amount of magnetic flux from the vertical stabilisation coils penetrating the outer limb, thereby taking into account the 3D distribution of the iron [65].

The iron currents calculated in the subroutine DOIRON are used as an initial guess in the EFIT fitting procedure. They are ascribed lower weights by EFIT than the measured shaping and feedback coil currents. On the other hand, the fields measured at the simulated probes are used only in the calculation of the iron currents and do not feature alongside the real magnetic probe measurements in the EFIT equilibrium solution. In sections 3.5.6–3.5.7 below the importance of the DOIRON model to the EFIT equilibrium results is assessed.
3.5.2 Permeability function

The iron current is a function of both the magnetic field and the permeability, which in turn is a function of the magnetic field. Though it is true that the iron structure contains regions of different iron materials with varying permeability characteristics (i.e. differing B-H curves), the DOIRON model assumes a uniform response function. It has been shown that taking different permeability regions into account does not substantially improve the agreement with experimental data [66]. Defining:

\[ X = |B| - B_{\text{saturated}} \]  \hspace{1cm} (3.10)

where \( B_{\text{saturated}} \) is the field at which the iron is fully magnetised. The prescribed function is:

\[ \frac{1}{\mu_r(B) - 1} = 0.025 + \frac{X + |X| + \log(1+e^{-10|X|})}{2B_{\text{saturated}}} \]  \hspace{1cm} (3.11)

Figure 3.8 shows the relative permeability as a function of the magnetic field.
field at the iron segment. The saturation magnetic field at 2.05 T marks a transition point in the function (notice $X + |X| = 0$ for $B < B_{\text{saturated}}$). $\mu_r$ stays relatively constant as $B \to 0$ for reasons of code stability. Also shown is the averaged $\mu_r(B)$ function used in the Proteus equilibrium code [67]. This curve is closer to the real $\mu_r(B)$ behaviour of the iron core.

3.5.3 EFUND Greens functions

The Greens function pertaining to every measurement used in EFIT is calculated by the standalone program EFUND. An input file mhdin describes the position, size and orientation of every element (e.g. Pickup coils, flux loops, saddle loops, shaping coils). The iron is modelled as 48 current carrying segments. Some of these segments are divided further for the purposes of calculating the Greens functions (into what will henceforth be known as 'sub-segments'). There are altogether 118 sub-segments forming the 48 segments. The parameter FCID in the input file serves to group sub-segments into their respective segments (see table 3.1).

| $R(m)$ | $Z(m)$ | $DR(m)$ | $DZ(m)$ | Angle1 | Angle2 | FCID | Seg
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>9.0150</td>
<td>0.87000</td>
<td>0.030</td>
<td>1.74</td>
<td>0.0</td>
<td>0.0</td>
<td>71.</td>
<td>/I1</td>
</tr>
<tr>
<td>9.0150</td>
<td>2.61000</td>
<td>0.030</td>
<td>1.74</td>
<td>0.0</td>
<td>0.0</td>
<td>72.</td>
<td>/I2</td>
</tr>
<tr>
<td>9.0150</td>
<td>4.35000</td>
<td>0.030</td>
<td>1.74</td>
<td>0.0</td>
<td>0.0</td>
<td>73.</td>
<td>/I3</td>
</tr>
<tr>
<td>1.8720</td>
<td>4.64416</td>
<td>0.144</td>
<td>0.03</td>
<td>0.0</td>
<td>0.0</td>
<td>74.</td>
<td>/I4</td>
</tr>
</tbody>
</table>

Table 3.1: Excerpt from the mhdin input file describing the iron. From left to right the numbers describe the $(R,Z)$ location of the iron segment centre; the segment width $(DR)$, height $(DZ)$ and orientation angles, and the FCID segment identification number.

A subroutine in EFUND known as BRBZV calculates the mutual inductance between two circular filaments of known radii and separation. This inductance is calculated for subdivided regions in a current element and integrated over the whole (in subroutine BTANFC), yielding the Greens function. The Greens function for each segment is then derived from its constituent sub-segment contributions. In EFUND, the iron segments are treated in the same way as the plasma shaping coils and their influence is contained in the same 'Greens function' arrays (RMP2FC, RSILFC etc).
3.5.4 Testing the iron model

In order to test the validity of the iron model, it is essential that the complicating presence of the unknown plasma current is absent; i.e. the model is tested using data from a dry run. In a typical dry run, the independent currents in the poloidal field and ohmic coils are switched on and off in turn. The iron magnetization currents can be calculated from the known currents using the DOIRON subroutine; in this case treated as a standalone program. The magnetic field at the probe positions is calculated based on the iron magnetization and the known currents. Thus, the accuracy of the iron model can be ascertained. It was found that, in general, the EFIT iron model is inaccurate in its prediction of the probe measurements [68] (see figure 3.9). It is of interest to investigate to what extent modifications to the iron model affect the EFIT results.

![Figure 3.9: Magnetic field measured (bold line) and predicted by the EFIT iron model at a poloidal pick-up coil for the duration of a dry run. The error bar represents two standard deviations of the measured data.](image)
Figure 3.10: Magnetic flux surface geometry recovered by EFIT where 48 (pink contours) and 32 (blue) segments are used.

3.5.5 Modifying the iron model

In order to make changes to the iron model, the input to EFUND must be modified in order that a new set of Green’s functions can be calculated. To expedite the process a program NEWIRON was written by this author. NEWIRON creates an input file, ‘mhdin_iron’, for input to EFUND. The iron geometry is specified in NEWIRON, at the crudest level, by a series of ‘corner’ points. These corner points define co-linear segments of the iron. Between each two neighbouring corner points the number of iron segments and sub-segments may be specified. The orientation of each iron segment is also input. The code assumes that the iron is up-down symmetric. Two simulated pickup coils are placed, one orientated normally and the other tangentially to the iron surface, close to each segment location. An IDL program (newiron.pro) can be used to plot the rearranged locations of the iron segments.

The newly calculated iron Greens functions may then be used by EFIT (following simple changes to the include file ‘eparves.inc’). DOIRON will
calculate the iron currents as before, though the location and number of those currents may have changed. Often, and for reasons as yet unknown, the DOIRON subroutine did not converge (i.e. via the algorithm described in section 3.5.1) to a solution for the iron currents when the iron model was altered. However it is possible to force EFIT to use the iron currents arrived at, after an arbitrary number of iterations, in order to solve the equilibrium.

![Current profile recovered by EFIT when the iron is simulated by 48, 46, 44, 36 and 32 segments.](image)

**Figure 3.11:** Current profile recovered by EFIT when the iron is simulated by 48, 46, 44, 36 and 32 segments.

### 3.5.6 Reducing the number of iron segments

This section examines whether reducing the number of iron segments in the iron model has a significant effect on the EFIT equilibrium solution. Since the iron magnetization currents are given a low weight ($\approx 0.01$) in the EFIT equilibrium solution, it might be expected that the effect of changes to the iron model on the EFIT results will be relatively small. To reduce the number of simulated iron currents, neighbouring segments were merged together. Merging of segments in the upper half of the iron results in a mirror image merging in the lower half to preserve up-down symmetry. The merged segments will act as one larger segment in the calculation of the iron currents in the DOIRON code. By merging co-linear sections of the iron together it was possible to reduce the number of segments to 32 while preserving the shape of the iron-air boundary. To reduce the number of iron segments below 32 it proved necessary to remove 'corner' points. This would result in a change to
the shape of the iron as modelled by DOIRON. The five segments in the gently sloping upper and lower section of the iron proved impossible to remove without EFIT crashing. These were preserved but the corner 'shoe' section was removed. The effect these changes to the iron model had on the EFIT calculation is surmised in table 3.2.

<table>
<thead>
<tr>
<th>Segment #</th>
<th>$\beta_{\text{pol}}$</th>
<th>$l_i$</th>
<th>$q_{\text{ax}}$</th>
<th>$R_{\text{mag}}$</th>
<th>$Z_{\text{mag}}$</th>
<th>$R_x$</th>
<th>$Z_x$</th>
<th>$\chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>48</td>
<td>0.377</td>
<td>0.810</td>
<td>1.622</td>
<td>2.994</td>
<td>0.326</td>
<td>2.663</td>
<td>-1.426</td>
<td>28.39</td>
</tr>
<tr>
<td>46</td>
<td>0.376</td>
<td>0.812</td>
<td>1.622</td>
<td>2.994</td>
<td>0.325</td>
<td>2.663</td>
<td>-1.427</td>
<td>28.42</td>
</tr>
<tr>
<td>44</td>
<td>0.380</td>
<td>0.804</td>
<td>1.638</td>
<td>2.995</td>
<td>0.325</td>
<td>2.663</td>
<td>-1.427</td>
<td>29.12</td>
</tr>
<tr>
<td>36</td>
<td>0.383</td>
<td>0.788</td>
<td>1.694</td>
<td>2.994</td>
<td>0.328</td>
<td>2.664</td>
<td>-1.428</td>
<td>30.65</td>
</tr>
<tr>
<td>32</td>
<td>0.384</td>
<td>0.781</td>
<td>1.715</td>
<td>2.994</td>
<td>0.329</td>
<td>2.664</td>
<td>-1.428</td>
<td>29.87</td>
</tr>
<tr>
<td>20</td>
<td>0.364</td>
<td>0.793</td>
<td>1.683</td>
<td>2.990</td>
<td>0.328</td>
<td>2.665</td>
<td>-1.427</td>
<td>31.28</td>
</tr>
</tbody>
</table>

Table 3.2: Some of the main plasma equilibrium characteristics calculated by EFIT when the iron was modelled with 48, 46, 44, 36, 32 and 20 segments. $\beta_{\text{pol}}$ is poloidal beta, $l_i$ is in the internal inductance, $q_{\text{ax}}$ is the axial safety factor, $(R_{\text{mag}},Z_{\text{mag}})$ is the magnetic axis location and $(R_x,Z_x)$ are the co-ordinates of the X-point. $\chi^2$ is the goodness of fit of the EFIT run. This data was obtained for pulse number 51675 and at time point 46.9s.

The plasma flux surfaces and the plasma current obtained when the iron geometry was left unchanged but the number of segments was reduced from 48 to 32 segments is shown in figures 3.10 and 3.11 respectively. It is clear that modifying the number of segments in this way has little effect on the magnetic geometry. In addition, it is apparent that the current density profile is relatively unaffected at the plasma boundary but becomes increasingly perturbed towards the magnetic axis. At its maximum, this change to the current density profile is always less than 5% (even in the 20 segment case where the iron geometry is also modified). It was found that the currents calculated by DOIRON were approximately up-down symmetric. The saturated inner iron limb is represented by segments 20-24 and 44-48 in the 48 segment case and by segments 16 and 32 in the 32 segment case. A current of 900kA flows in each segment in the 48 segment case, a total of 9MA. In the 32 segment case each segment carries 4.5MA. Thus, the current DOIRON calculates on the inner limb of the iron structure is independent of the number of segments.

It is interesting to note the effect that changing the number of iron segments has on the calculated contribution (due to the plasma and iron currents) to the field at the magnetic pickup coils. There are some small changes
to the iron contribution while the plasma contribution remains virtually unchanged when number of iron segments is reduced from 48 to 20. For example, figure 3.12 examines the iron and plasma current influence on pickup coil 12 over time. Although the field due to the iron currents at this pickup coil changes by as much as 0.01T, the total field due to all currents changes by less than half this amount. As evident in figure 3.13(a) this is largely due to compensatory changes in the calculated shaping coil currents. The goodness of fit in the 20 segment case is consistently worse than the 48 segment case (fig. 3.13(b)), the former being as much as 40% worse than the latter.

Figure 3.12: Time evolution of field calculated at pickup coil 12 due to plasma current (top), iron currents (middle) and all currents (bottom) where 48 (solid line) and 20 (dash) segments were used to model the iron.

### 3.5.7 Changing the iron geometry

The previous section was concerned with reducing the number of segments while keeping fixed, insofar as possible, the position of the iron-air boundary. This boundary is derived from an equivalent 2D axisymmetric model described in [65]. An alternative equivalent model, given in [66], follows the
3.6 Conclusion

Changes in the iron currents and the iron Greens functions have little effect on the EFIT equilibrium results. The results are robust to changes in the iron model effected by reducing the number of segments, moving the iron boundary or reducing the number of iterations in DOIRON. This robustness...
simply reflects one of the advantages of solving the magnetic equilibrium over a full domain - that the unknown currents (which could also include vessel and halo currents) are accommodated by allowing the magnetic measurements to vary within their experimental uncertainty when solving the Grad-Shafranov equation. The variation in the current density near the magnetic axis (figure 4) due to reducing the number of iron segments is within the established EFIT uncertainty. When testing the iron model using dry runs, it was found that it is inaccurate in predicting the magnetic field at the magnetic probe locations. However, the results presented here suggest that such inaccuracy need not significantly compromise the results of an interpretative equilibrium such as EFIT given a sufficient number of magnetic measurements.
Chapter 4

TRANSP transport code

4.1 Introduction

TRANSP is a time dependent tokamak transport data analysis code. It evolved from the BALDUR code in 1977, Goldston added the Monte Carlo neutral beam package to the code in 1979. TRANSP was originally developed at PPPL and is now in use at a number of tokamaks around the world; it was installed at JET in 1987. The modus operandus of the code is to make maximal use of diagnostic data to infer the transport and confinement properties of plasmas. A key strength of TRANSP is that the consistency checks it provides can be used to check the accuracy of JET experimental data or various analytical models. TRANSP may also be used as a predictive tool, notably in the study of the current profile. The breadth of physics included in the code, which will be touched on in this chapter, allows the study of numerous aspects of plasma physics including particle transport and magnetic field diffusion.

The main modules of the TRANSP code, illustrated in figure 4.1, form the basis of the sectional form of this chapter. Section 4.2 describes the input data supplied to the code. In section 4.3, how TRANSP calculates the magnetic geometry of the plasma is discussed. Section 4.4 is concerned with the particle transport model. Section 4.5 deals with particle balance and energy balance equations in broad terms. Finally, section 4.6 is concerned with the magnetic diffusion equation. This last section is the most comprehensive since the main findings of this thesis are pursuant to it.
4.2 Input Data

A full analytical TRANSP run utilizes a wide range of input data. The minimum set of data consist of the electron density and temperature, the surface voltage, Z-effective, the radiated power and the plasma current. Where possible the charge-exchange spectroscopy (CXS) derived ion parameters (namely temperature, density and angular velocity) are input. Such quantities as gas recycling rates, sawtooth crash times and the safety factor profile may also be supplied. This section will describe the main experimental data used in TRANSP, focusing on those particularly relevant to the subject of this thesis.

4.2.1 Electron temperature and density

The electron temperature profile is required to be input in any TRANSP run. It is used to calculate the resistivity (essential in solving the magnetic
diffusion equation), in the neutral source and particle balance calculation and also to evaluate the electron and ion energy balance equations. Where possible the temperature measured by Electron Cyclotron Emission (ECE) is used in preference to the LIDAR Thomson scattering data due to its superior spatial and temporal resolution. A Michelson interferometer\(^1\) measures the ECE spectrum in the extraordinary mode. The optically thick second harmonic region is used to find the electron temperature profile. The radial resolution of the diagnostic is about 4 cm. The spectral scan rate can range from 4 to 60 scans a second; the duration of each scan is the inverse of the rate. The temperature data may either be input in its raw form against frequency (ECM1/CSPC) or alternative mapped onto major radius (ECM1/PRFL). This mapping may be carried out by the TRANSP utility program 'ECECON', in concert with the internal equilibrium code VMEC. This is considered more sensible than inputting the externally mapped ECE electron temperature (which is done using the FAST magnetic flux geometry). The normal level of accuracy of the ECE system is 10%. However, the ECE temperature is inaccurate in the presence of suprathermal electrons. Thus, it is generally unsound in pulses involving lower hybrid heating. The data should also be treated with care in the presence of ELMs or other substantial changes in \(T_e\). In this case the LIDAR Thomson scattering electron temperature (LIDR/TE) is used.

Thomson scattering is the scattering of electromagnetic waves by free electrons. The electrons will re-radiate the incident waves at a Doppler shifted frequency. The frequency of light scattered by a Maxwellian ensemble is described by a Gaussian centred at the incident frequency (ignoring relativistic effects). The width of this Gaussian is a measure of the velocity and hence the temperature of the electrons. The area under the curve is related to the electron density. In the LIDAR system the frequency of the back-scattered light is measured [69]. This time-of-flight technique greatly simplifies the required collection optics (as opposed to the 90\(^\circ\) Thomson scattering technique) [70].

The core LIDAR system (KE3 in JET terminology) consists of a 4 Hz, 1 J, 300 ps ruby laser (\(\lambda = 694.3\text{nm}\)) and measures the electron temperature and density. The line of sight of this diagnostic is almost radial but inclined at 2.8 degrees from the major radius. The resolution of the diagnostic along the line of sight is 12 cm. An upper and lower limit on the temperature is

\(^1\)A SPECAC fast scanning interferometer is used at JET.
also calculated (LIDR/TEU and LIDR/TEL respectively). In general, the LIDAR electron temperature profile is more noisy than that measured by ECE. When smoothing the data, care should be taken that the effects on the temperature measurements due to events such as sawteeth should be preserved.

The electron density is essential in a number of calculations in TRANSP such as in the determination of the impurity density, ion and electron energy balance, pressure profile and neutral source. The most accurate measure of the electron density is provided by the Far-Infrared (FIR) interferometer. However interferometry is restricted to a few channels, and so core LIDAR is used to provide profile information. The LIDAR electron density measurements are periodically calibrated against the interferometer data.

### 4.2.2 Ion temperature and angular velocity

TRANSP can operate in either ion temperature predictive or analysis mode. Generally a TRANSP run will consist of a predictive phase preceding an analysis phase. In predictive mode, the ion power balance equation is solved with the ion temperature as the unknown. In this case $\chi_i$, the ion thermal conductivity may be set to some multiple of the electron thermal conductivity $\chi_e$. In interpretive mode one option is to vary the ion thermal conductivity so that the measured neutron yield is matched.

The ion temperature is measured by a number of diagnostics including charge exchange, high resolution X-ray and neutron spectroscopy. Active charge exchange, possible only when neutral beams are heating the plasma, provides the most detailed and direct measurement of ion temperature, angular velocity and impurity content [38], [71]. In the core of the plasma where temperatures typically exceed 20keV, all of the low-Z impurity ions are fully stripped. Passive line emission due to partially ionised atoms at the plasma edge (where the temperature drops below 5keV) will only reveal information about that region. By injecting fast neutrals into the core of the plasma, a transfer of charge may take place between the incoming neutrals and the resident impurity ions.

$$A^z + D^- \rightarrow (A^{z+1})^* + D^+$$  \hspace{1cm} (4.1)

The excited and newly ionised impurity ion will then radiate at specific wave-
lengths as it de-excites. Since optical fibres are used to collect the emitted light, only transitions that emit in the visible range may be observed. A beam-splitter is used so that light from several impurities can simultaneously be analysed. A spectrometer measures a full spectrum about every 20ms which is then resolved using a multi-Gaussian fit. Contributions due to other processes such as direct electron excitation of the excited ion must often be subtracted.

The plasma rotation is inferred from the amount of Doppler shifting in the charge spectrum while the ion temperature is found from the broadening of the spectral lines. Due to the lines of sight of the charge exchange diagnostic intersecting the neutral beam paths at multiple locations in the plasma, profiles of ion temperature, angular velocity and Z-effective are obtained. At high beam energies and plasma temperatures the lineshapes in the charge exchange spectrum form a distorted Gaussian due to the fact that the charge exchange cross-section are greater for the plasma ions moving with parallel to the beam rather than anti-parallel. The code CHEAP produces corrected charge exchange results which are used in the TRANSP run where available [72].

4.2.3 Z-effective

The effective ionic charge of the plasma, $Z_{\text{eff}}$, is used in calculating the electrical resistivity, impurity concentration and hydrogenic ion depletion factor of the plasma. In TRANSP, the plasma composition and resistivity $Z_{\text{eff}}$ are distinct and not necessarily the same. This allows the rate of magnetic field diffusion to be determined by some other constraint while leaving the plasma composition $Z_{\text{eff}}$ unchanged. For example, the user may stipulate that TRANSP matches measured loop voltage. TRANSP may do this by varying the resistivity $Z_{\text{eff}}$ only.

The total measured plasma current $I_p$ can be conserved during a TRANSP run by using $Z_{\text{eff}}$ data together with electron temperature data to find the resistivity but allowing the loop voltage $V_l$ to float (i.e. it is a predicted quantity). Alternatively, both $I_p$ and $V_l$ can be conserved by allowing $Z_{\text{eff}}$ to float.

A line of sight averaged $Z_{\text{eff}}$ is obtained using the visible Bremsstrahlung
measurement which measures the following integral

\[ I(l) = g(l) \int_{LOS} \frac{Z_{\text{eff}} n_e^2}{T_e^3} \]  

(4.2)

where \( g(l) \) is the calibration function. Utilising the electron density determined by FIR interferometer and the temperature found by LIDAR Thomson scattering allows \( Z_{\text{eff}} \) to be obtained. A line-averaged measure of \( Z_{\text{eff}} \) is found along a vertical and horizontal chord is found. The difference between the two quantities amounts to a measure of the accuracy of the data. A \( Z_{\text{eff}} \) profile may be available during the neutral beam switch on times from charge exchange spectroscopy.

### 4.2.4 Plasma Heating

JET plasmas are heated ohmically, by neutral beams and by RF waves (ion cyclotron resonance heating (ICRH) and lower hybrid current drive (LHCD)). TRANSP runs must start during the ohmic phase before beams or RF power switch on. Unfortunately at JET, there is no working LHCD module in the TRANSP code. The neutral beam parameters are automatically included in the TRANSP namelist by use of a script. ICRH parameters, such as the phase of waves being launched from each antenna, may be manually entered into the namelist. The effect of neutral beams on the plasma is modelled using Monte Carlo techniques (in the package NUBEAM, see next section for details).

### 4.2.5 Loop voltage and Neutron yield

These two quantities are usually used as consistency checks when examining the TRANSP output. If the calculated loop voltage deviates significantly from that measured it may be due to inaccuracies in the edge electron temperature measurement. TRANSP calculates neutrons due to plasma-plasma, beam-plasma and beam-beam reactions. An inconsistent prediction of neutron yield can indicate inaccuracies in the ion temperature, density of \( Z_{\text{eff}} \) or density in the plasma core. In addition, both quantities may be used as constraints in the TRANSP run. The neutron yield may be used as an analytical tool in setting the ion thermal neoclassical multiplier. For an example of loop voltage and neutron yield consistency checks see figures 4.2 and 4.3.
4.3 Plasma magnetic geometry

TRANSP uses a full 2D MHD equilibrium solver (generally VMEC, the variational moments equilibrium code) to generate the internal plasma geometry (i.e. the flux surfaces). VMEC allows non-circular and up-down asymmetric magnetic flux surfaces. The plasma boundary, external magnetic field, the $q$ (safety factor) profile and pressure profile are all supplied to the equilibrium solver. Useful flux surface differential volume averaged quantities such as $\left\langle \frac{1}{R} \right\rangle_v$, where $R$ is the major radius, are calculated. The equilibrium solver also outputs the 'G' function, that is the para/diamagnetic correction to the toroidal field due to poloidal plasma current.

Since VMEC is a fixed boundary code, the plasma boundary must be supplied as input data. The code then operates within this specified boundary and takes no direct account of magnetic field or flux measurements provided by pickup coils and flux loops. However this information may implicitly be supplied by use of the EFIT (the free-boundary equilibrium code) plasma boundary. The PPF’s EFIT/RBND and EFIT/ZBND describe the bound-
Particle transport

4.4 Particle transport

TRANSP uses two separate packages to deal with thermal neutral and fast ion transport in a tokamak. FRANTIC is a fast 1d transport code which calculates the former. NUBEAM is a Monte Carlo package whose remit is the fast ions. Since fast ions are lost via charge exchange to the thermal population (and in so doing pass from the jurisdiction of NUBEAM to be-
come a source neutral in FRANTIC) and since the rate of this conversion is
dependent on knowing the neutral density (calculated by FRANTIC) these
two packages are mutually dependent.

FRANTIC takes into account 'edge' sources from recycling or gas flow,
and 'volume' sources due to recombination and charge exchange. It assumes
that each source term is a flux function. The code also takes into account the
depletion of neutrals due to impact ionisation or charge exchange with the
fast ions. For each neutral species the code calculates density, temperature
and toroidal velocity profiles.

In NUBEAM [73], a Monte Carlo simulation is used to calculate beam
deposition, fast ion orbits, charged particle collisions, charge exchange trans-
port of beam particles, beam driven currents and momentum transfer. It is
the most time expensive part of the code. Typically 1000 neutrals are tracked
by Monte Carlo, each one may undergo several atomic processes before being
lost to the bulk plasma (an ion is considered thermalised if its energy falls to
\( \frac{3}{2}T_i \) or less), to loss orbits or to charge exchange. 'Russian roulette’ is used
to maintain a constant census of particles. For each Monte Carlo particle,
whose energy and trajectory are randomly selected from the allowable range,
the following integral

\[
I = \int n_e \sigma_{tot} dl
\]

(4.3)
is calculated. Where the condition \( I = \ln(1/\eta) \), where \( \eta \) is a random number
is the range \( 0 < \eta < 1 \), is satisfied the particle is assumed ionised and
is then treated as a fast ion. The code accounts for volume effects in its
calculation of beam deposition. The beam stopping cross section \( \sigma_{tot} \) includes
contributions from impact ionisations with electrons, bulk hydrogenic bulk
ions and circulating beam ions, from charge exchange with hydrogenic bulk
ions and beam ions, and from total electron loss in collisions with impurities.

The newly deposited fast ions are tracked until they thermalise by inte-
grating simplified guiding centre orbit equations. The Fokker-Planck (F-P)
equation deals with the affect on a plasma species distribution function of
small angle charged particle collisions. The energy diffusion and pitch angle
spread that a Monte Carlo particle suffers may be simulated using the F-P
equation. This collisional jolt in velocity and pitch angle space is applied to
the Monte Carlo particle several times per orbit. Charge exchange events
are similarly included, using the appropriate collision operator. A speedier
but less comprehensive alternative to Monte Carlo simulation exists, that is
available in TRANSP, is to numerically solve the drift kinetic equation for fast ions using the F-P equation. The NUBEAM package also follows the fast alphas which result from fusion reactions.

4.5 Particle and Energy Balance

The evolution of the distribution function $f_\alpha(x, v, t)$ of a plasma species $\alpha$ is given by the Vlasov equation.

$$\frac{\partial f_\alpha}{\partial t} + v \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha}(E + \frac{v \times B}{c}) = C_\alpha + S_{f_\alpha}$$  \hspace{1cm} (4.4)

where $C_\alpha$ and $S_{f_\alpha}$ are the collisional and source terms respectively. The fluid equations are obtained by taking velocity moments of the Vlasov equation. The zeroth moment yields the particle balance equation.

$$\frac{\partial n_\alpha}{\partial t} + \nabla . (n_\alpha V_\alpha) = S_\alpha$$  \hspace{1cm} (4.5)

where the $n_\alpha$ and $V_\alpha$ are the fluid density and velocity respectively. The second term is the divergence of the flux ($\Gamma = \int f_\alpha v dv$) and since the source term is due to beam deposition, edge recycling or gas flow we may rewrite the particle balance equation.

$$\frac{\partial n_\alpha}{\partial t} = S_{NBI} + S_{gas} + S_{recycling} - \nabla \Gamma_\alpha$$  \hspace{1cm} (4.6)

The user can choose the model used to characterise the rate of change of density $\frac{\partial n_\alpha}{\partial t}$ and the ion outflow $\nabla \Gamma_\alpha$ (the source terms are usually input from experimental data). The models evolve ion species densities in such a way that $Z_{\text{eff}}$ and quasineutrality are honoured. Namely,

$$n_e = \sum Z_\alpha * n_\alpha$$  \hspace{1cm} (4.7)

$$n_e * Z_{\text{eff}} = \sum Z_\alpha^2 * n_\alpha$$  \hspace{1cm} (4.8)

As an initial condition to the partial differential equations that are solved numerically by each particle balance model, the initial concentrations of each thermal ion species, relative to one another, must be specified in the namelist. The main outputs of the particle balance model are the particle flux and diffusivity.
Having calculated the ion and electron source functions and particle flux via the particle transport and particle balance models respectively, TRANSP then calculates the ion temperature by solving the ion energy balance equation. The energy balance equation is obtained by taking the second moment of the Vlasov equation (i.e. multiply by \(mv^2\)). By making suitable substitutions for certain terms in the energy balance equation from the particle and momentum balance equations the thermal energy balance equation may be written.

\[
\frac{3}{2} \frac{\partial p_\alpha}{\partial t} + \nabla \cdot (q_\alpha + \frac{5}{2} p_\alpha V_\alpha) = -\Pi \nabla V_\alpha + V_\alpha \nabla p_\alpha + \sum_j Q_{\alpha j} + S_E \quad (4.9)
\]

where \(q_\alpha\) is the heat flux vector, \(\Pi\) is the stress tensor and \(Q_{\alpha j}\) represents the energy gained by species \(\alpha\) in collisions with species \(j\). TRANSP uses the ion power balance equation to compute the ion-electron coupling power \(Q_{ie} \approx n_e v_{ei} (T_e - T_i)\) and hence the ion temperature. Finally, the electron power balance equation is solved in order to compute the electron thermal conductivity \(\chi_e\) (the electron temperature is assumed to be a known quantity).

4.6 Magnetic field diffusion in TRANSP

4.6.1 TRANSP flux grid

In order to describe the TRANSP implementation of the magnetic field diffusion equation (described in section 2.4), a description of the magnetic flux grid upon which it is solved is necessary. The grid chosen is one which deforms with the plasma boundary such that the range of the flux label is time-invariant. This choice of grid is appropriate for TRANSP since all equations are solved within a prescribed plasma boundary.

Defining a relative flux label, \(\xi\), facilitates the use of a fixed set of grid points in the code.

\[
\xi = \sqrt{\frac{\phi(\rho)}{\phi_{\text{bnd}}}} = \frac{\rho}{\rho_{\text{bnd}}} \quad (4.10)
\]

where \(\phi_{\text{bnd}}\) is the total toroidal flux enclosed in the plasma and \(\phi(\rho)\) is the
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4.6 Magnetic field diffusion in TRANSP

toroidal flux, a function of the absolute flux label $\rho$ given by

$$\rho = \sqrt{\frac{\phi(\rho)}{\pi \langle B_o \rangle}}$$ (4.11)

where $\langle B_o \rangle$ is the area averaged toroidal magnetic field (note that $\rho$ is a flux surface radius, in metres, close to but not equivalent to the geometric radius of a flux surface). It follows that since $\phi_{bnd} = \pi \rho_{bnd}^2 \langle B_o \rangle$, the relative flux label $\xi$ is independent of $\langle B_o \rangle$. It is evident that the range $0 < \xi < 1$ covers the plasma at all time points, whatever the enclosed toroidal flux. Within the TRANSP code the calculations are carried out on the $\xi$ grid but output on the $\rho$ grid. Any time derivative carried out at a location defined by the absolute flux surface label $\rho$ can be expressed in terms of a relative flux label $\xi$ as follows

$$df[\xi,t] = \frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial t} dt$$ (4.12)

$$= \frac{\partial f}{\partial \xi} \left( \frac{\rho}{\rho_{bnd}} \right) + \frac{\partial f}{\partial t} dt$$ (4.13)

$$= \frac{\partial f}{\partial \xi} \left( \frac{d\rho}{\rho_{bnd}} - \frac{\rho}{\rho_{bnd}^2} d\rho_{bnd} \right) + \frac{\partial f}{\partial t} dt$$ (4.14)

and thus on a $\rho$-grid one has

$$\frac{\partial f}{\partial t} |_{\rho-grid} = -\frac{\partial f}{\partial \xi} \frac{\rho}{\rho_{bnd}} \frac{\partial \rho_{bnd}}{\partial t} + \frac{\partial f}{\partial t}$$ (4.15)

and hence

$$\frac{\partial f}{\partial t} |_{\rho-grid} = \frac{\partial}{\partial t} |_{\xi-grid} - \xi \kappa \frac{\partial}{\partial \xi}$$ (4.16)

where $\kappa = \frac{1}{\rho_{bnd}} \frac{\partial \rho_{bnd}}{\partial t}$.

Since the gradient of the absolute flux surface label $\rho$ is perpendicular to the local flux surface, it follows that the local perpendicular distance $\Delta h$ between two neighbouring flux surfaces with labels $\rho$ and $\rho + \Delta \rho$ is given by $\Delta h = \frac{\Delta \rho}{|\nabla \rho|}$. The volume $\Delta V$ bounded by two surfaces is

$$\Delta V = \oint dl_\theta \Delta h \times 2\pi R$$ (4.17)

$$= 2\pi \Delta \rho \oint dl_\theta \frac{R}{|\nabla \rho|}$$ (4.18)
where $dl_\theta$ is a infinitesimal length of arc along a flux surface, $V$ is the enclosed volume and $R$ is the major radius. In the limit $\Delta \rho \to 0$ we obtain the differential volume element

$$\frac{dV}{d\rho} = 2\pi \int dl_\theta \frac{R}{|\nabla \rho|} \quad (4.19)$$

One may then introduce the flux surface differential volume average of a function $f$ which subsequently proves useful

$$<f>_v = 2\pi (\frac{dV}{d\rho})^{-1} \int dl_\theta \frac{Rf}{|\nabla \rho|} \quad (4.20)$$

![Figure 4.4: Adjacent flux surfaces as labelled by TRANSP](image)

### 4.6.2 Magnetic field diffusion equation on $\xi$ grid

To see how the magnetic diffusion equation is implemented in the code it is necessary to express it in terms of flux surface differential volume averaged quantities (equation 4.20); these quantities are mapped onto the $\xi$ grid. The three laws upon whose foundations the magnetic diffusion equation rests, namely those of Faraday, Ampère and Ohm, will first be expressed in these terms. The derivation which follows is largely derived from ref. [74].
Faraday’s Law

Beginning with Faraday’s law, it is worthwhile to first note that the loop voltage is defined as follows.

\[ V_l = \oint \mathbf{E} \cdot d\mathbf{r} = 2\pi RE_\phi = -2\pi \frac{\partial \psi}{\partial t} \quad (4.21) \]

where \( \mathbf{E} \) is the electric field vector and the integration is carried around one toroidal circuit. The \( 2\pi \) factor is necessary as \( \psi \) is the poloidal flux per radian. In this subsection the loop voltage will be expressed in terms of the flux surface differential volume averaged quantity \( \langle \mathbf{E} \cdot \mathbf{B} \rangle_v \). In toroidal and poloidal components we have

\[ \langle \mathbf{E} \cdot \mathbf{B} \rangle_v = \langle E_\phi B_\phi \rangle_v + \langle E_\theta B_\theta \rangle_v \quad (4.22) \]

Note that \( B_\theta = \frac{1}{R} |\nabla \psi| \) can be expressed as \( B_\theta = \frac{1}{R} \frac{\partial \psi}{\partial \rho} |\nabla \rho| \). Then equation 4.22 can be written as

\[ \langle \mathbf{E} \cdot \mathbf{B} \rangle_v = \left( \frac{V_l}{2\pi R} \langle RB_\phi \rangle_v \right) + \left( \frac{E_\theta}{R} \langle \nabla \rho \frac{\partial \psi}{\partial \rho} \rangle_v \right) \quad (4.23) \]

The quantities \( RB_\phi, V_l \) and \( \frac{\partial \psi}{\partial \rho} \) are flux surface constants and so equation 4.23 may be simplified to (applying eq. 4.20)

\[ \langle \mathbf{E} \cdot \mathbf{B} \rangle_v = \left( \frac{RB_\phi}{2\pi} \langle \frac{1}{R^2} \rangle_v \right) + \left( \frac{E_\theta}{R} \langle \nabla \rho \frac{\partial \psi}{\partial \rho} \rangle_v \right) \quad (4.24) \]

\[ = \left( \frac{RB_\phi}{2\pi} \langle \frac{1}{R^2} \rangle_v \right) + \left( \frac{\partial \psi}{\partial \rho} \right) \left( \frac{2\pi}{\nabla \rho} \right) \oint dl_\theta E_\theta \quad (4.25) \]

The second term on the right hand side of eqn. 4.25 contains the integral \( \oint dl_\theta E_\theta \) which, by Lenz’s Law, is equal to \(- \frac{\partial \phi}{\partial t}\) (i.e. it defines a poloidal loop voltage). As the set of flux surfaces is defined by the toroidal flux, it follows that on any given flux surface \( \frac{\partial \phi}{\partial t} \) is zero. Rewriting this equation in terms of the loop voltage we have

\[ V_l = \frac{2\pi \langle \mathbf{E} \cdot \mathbf{B} \rangle_v \langle \frac{1}{R^2} \rangle_v}{(RB_\phi)} \quad (4.26) \]
It remains to substitute this expression for the loop voltage into Faraday’s law (eqn 4.21) to obtain the following expression for Faraday’s law.

$$-\frac{\partial \psi}{\partial t} = \frac{(E \cdot B)_v}{(RB_\phi) \left( \frac{1}{R^2} \right)_v}$$  \hspace{1cm} (4.27)

It proves convenient later in the derivation, when casting the magnetic field diffusion equation in terms of $\iota$, to differentiate both sides of Faraday’s law with respect to $\rho$ (see equations 4.49 and 4.53).

$$-\frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial \rho} \right) = \frac{\partial}{\partial \rho} \left[ \frac{(E \cdot B)_v}{(RB_\phi) \left( \frac{1}{R^2} \right)_v} \right]$$  \hspace{1cm} (4.28)

**Ampère’s Law**

In the case of expressing Ampere’s law in terms of flux surface quantities it is best to first separately derive expressions for the toroidal and poloidal current densities ($J_\phi$ and $J_\theta$). Ampere’s law in integral form for the toroidal current is

$$\oint B_\theta \cdot dl_\theta = \mu I_\phi$$  \hspace{1cm} (4.29)

Noting that $|B_\theta| = \frac{1}{R} |\nabla \psi| = \frac{1}{R} \frac{\partial \psi}{\partial \rho} |\nabla \rho|$ and that $\frac{\partial \psi}{\partial \rho}$ is a flux surface quantity the expression for the toroidal current becomes

$$\frac{\partial \psi}{\partial \rho} \oint |\nabla \rho| \frac{1}{R} dl_\theta = \mu I_\phi$$  \hspace{1cm} (4.30)

Using equation 4.20 one has $< \frac{\nabla \rho^2}{R^2} >_v = 2\pi \left( \frac{dV}{d\rho} \right)^{-1} \oint |\nabla \rho| dl_\rho$. The toroidal current within a flux surface $\rho$ may now be written in terms of flux surface quantities.

$$I_\phi(\rho) = \frac{1}{2\pi \mu} \left( \frac{dV}{d\rho} \right) \left( \frac{\partial \psi}{\partial \rho} \right) < \frac{\nabla \rho^2}{R^2} >_v$$  \hspace{1cm} (4.31)

The toroidal current within an arbitrary flux surface may be expressed in terms of the toroidal current density $J_\phi$ by integrating over the constituent differential area elements (see figure 4.4).

$$I_\phi(r) = \int_r \oint d\theta dh J_\phi$$  \hspace{1cm} (4.32)
Since $dh = \frac{d\rho}{|\nabla \rho|}$ one may write

$$I_\phi(r) = \int_\rho \int_\theta \, d\theta \, d\rho \frac{d\rho}{|\nabla \rho|} I_\phi$$

(4.33)

and it follows

$$\frac{\partial}{\partial \rho} I_\phi(\rho) = \oint d\theta |\nabla \rho| J_\phi$$

(4.34)

From equation 4.20 we have

$$\left\langle \frac{J_\phi}{R} \right\rangle_v = 2\pi \left( \frac{dV}{d\rho} \right)^{-1} \oint d\theta |\nabla \rho| J_\phi$$

(4.35)

Equations 4.34 and 4.35 may be combined into reading

$$\left\langle \frac{J_\phi}{R} \right\rangle_v = 2\pi \left( \frac{dV}{d\rho} \right)^{-1} \frac{\partial}{\partial \rho} I_\phi$$

(4.36)

and so, by substitution of equation 4.31 for $I_\phi$ into the above expression, one has the following relation for the toroidal current density conveniently expressed in terms of flux surface quantities.

$$\left\langle \frac{J_\phi}{R} \right\rangle_v = \frac{1}{\mu} \left( \frac{dV}{d\rho} \right)^{-1} \frac{\partial}{\partial \rho} \left[ \left( \frac{dV}{d\rho} \right) \left( \frac{\partial \psi}{\partial \rho} \right) \left\langle \frac{|\nabla \rho|^2}{R^2} \right\rangle_v \right]$$

(4.37)

Ampère’s law is again employed to obtain a relation for the poloidal current density $J_\theta$. Taking any poloidal cut through the plasma,

$$\mu I_\theta = \oint \mathbf{B} \cdot d\mathbf{l}$$

(4.38)

$$= \oint B_\phi R d\phi$$

(4.39)

$$= B_\phi R \oint d\phi$$

(4.40)

$$= 2\pi RB_\phi$$

(4.41)

and it proves useful to differentiate both sides with respect to $\rho$

$$\mu \frac{\partial I_\theta}{\partial \rho} = 2\pi \frac{\partial (RB_\phi)}{\partial \rho}$$

(4.42)
Since the poloidal current in a differential toroidal annulus is given by
\[ dI_\theta = 2\pi R J_\theta dh, \]
and since \( dh = \frac{d\rho}{|\nabla \rho|} \), we can write
\[ \frac{\partial I_\theta}{\partial \rho} = \frac{2\pi R}{|\nabla \rho|} J_\theta \] (4.43)

By combining equations 4.42 and 4.43, the following expression for the poloidal current density follows
\[ |J_\theta| = -\frac{1}{\mu} \frac{|\nabla \rho|}{R} \frac{\partial (RB_\phi)}{\partial \rho} \] (4.44)

It then remains to merge the two contributions to the current together to find an expression for the “parallel” Ampere’s Law. This is found by using \( \langle J.B \rangle_v = \langle J_\phi B_\phi \rangle_v + \langle J_\theta B_\theta \rangle_v \). Noting that \( B_\theta = \frac{1}{R} \nabla \psi = \frac{1}{R} \frac{\partial \psi}{\partial \rho} \nabla \rho \) and that the the quantities \( RB_\phi \) and \( \frac{\partial \psi}{\partial \rho} \) are flux surface quantities one may write
\[ \langle J.B \rangle_v = (RB_\phi) \langle \frac{J_\phi}{R} \rangle_v + \frac{\partial \psi}{\partial \rho} \langle J_\theta \nabla \rho \rangle_v \] (4.45)

Substituting equations 4.37 and 4.44 into the above yields the following expression.
\[ \langle J.B \rangle_v = \frac{(RB_\phi)}{\mu} \left( \frac{dV}{d\rho} \right)^{-1} \frac{\partial}{\partial \rho} \left( \frac{dV}{d\rho} \right) \left( \frac{\partial \psi}{\partial \rho} \right) \left\langle \frac{|\nabla \rho|^2}{R^2} \right\rangle_v + \right. \]
\[ -\frac{1}{\mu} \frac{\partial \psi}{\partial \rho} \frac{\partial (RB_\phi)}{\partial \rho} \left\langle \frac{|\nabla \rho|^2}{R^2} \right\rangle_v \] (4.46)

Finally since we have,
\[ \frac{\partial}{\partial \rho} \left[ \left( \frac{dV}{d\rho} \right) \left( \frac{\partial \psi}{\partial \rho} \right) \frac{1}{(RB_\phi)} \left\langle \frac{|\nabla \rho|^2}{R^2} \right\rangle_v \right] = \frac{1}{RB_\phi} \frac{\partial}{\partial \rho} \left[ \left( \frac{dV}{d\rho} \right) \left( \frac{\partial \psi}{\partial \rho} \right) \left\langle \frac{|\nabla \rho|^2}{R^2} \right\rangle_v \right] + \right. \]
\[ -\frac{1}{(RB_\phi)^2} \left( \frac{dV}{d\rho} \right) \left( \frac{\partial \psi}{\partial \rho} \right) \left( \frac{\partial (RB_\phi)}{\partial \rho} \right) \left\langle \frac{|\nabla \rho|^2}{R^2} \right\rangle_v \]

the terms in equation 4.46 may be amalgamated into the following equation which neatly expresses the current density in terms of flux surface quantities.
\[ \langle J.B \rangle_v = \frac{1}{\mu} \left( \frac{dV}{d\rho} \right)^{-1} (RB_\phi)^2 \frac{\partial}{\partial \rho} \left[ \left( \frac{dV}{d\rho} \right) \left( \frac{\partial \psi}{\partial \rho} \right) (RB_\phi)^{-1} \left\langle \frac{|\nabla \rho|^2}{R^2} \right\rangle_v \right] \] (4.47)
Ohm’s Law and solution

Ohm’s law relates \( \langle E \cdot B \rangle_v \) to \( \langle J \cdot B \rangle_v \), less any non-inductively driven currents.

\[
\langle E \cdot B \rangle = \eta_\parallel [\langle J \cdot B \rangle - \langle J \cdot B \rangle_{N1}] 
\]

where \( \eta_\parallel \) is the resistivity parallel to the magnetic field (see section 2.4.3).

To obtain an expression for the magnetic field diffusion equation simply substitute Ampère’s law (eqn. 4.47) into Ohm’s Law 4.48, and the resulting expression into Faraday’s law (eqn. 4.28). The equation which results follows

\[
\frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial \rho} \right) = \frac{\partial}{\partial \rho} \left( \eta_\parallel \left( \frac{dV}{d\rho} \right)^{-1} (RB_\phi) \frac{\partial}{\partial \rho} \left[ dV \frac{\partial \psi}{d\rho} \frac{|\nabla \rho|^2}{R^2} \right] \right) + \\
- \frac{\partial}{\partial \rho} \left[ \frac{\eta_\parallel \langle J \cdot B \rangle_{NI}}{(RB_T) \langle R^{-2} \rangle} \right] 
\]

This expression must be transformed onto the fixed \( \xi \) grid. A time derivative of some quantity on the \( \rho \) grid can be transformed to one on the \( \xi \) grid using eqn. 4.16. The following relations are used in the transformation onto the \( \xi \) grid: \( \rho = \rho_{bnd} \xi \), \( \frac{\partial}{\partial \rho} = \frac{1}{\rho_{bnd}} \frac{\partial}{\partial \xi} \) and \( t = \frac{2\pi \partial \psi}{\partial \phi}. \) In fact TRANSP advances the iota-bar profile in time. An expression relating \( \frac{\partial \psi}{\partial \rho} \) and \( \iota \) is derived as follows

\[
\frac{\partial \psi}{\partial \rho} = \left( \frac{\partial \psi}{\partial \phi} \right) \left( \frac{\partial \phi}{\partial \rho} \right) 
\]

\[
= \left( \frac{\iota}{2\pi} \right) \left( \frac{2\rho \Phi_{bnd}}{\rho_{bnd}} \right) 
\]

\[
= \frac{\iota \xi \Phi_{bnd}}{\pi \rho_{bnd}} 
\]

where \( \Phi_{bnd} \) is the total enclosed toroidal flux. The previously introduced quantity \( \kappa = \frac{1}{\rho_{bnd}} \frac{\partial \Phi_{bnd}}{\partial \rho} = \frac{1}{2\pi \rho_{bnd}} \frac{\partial \Phi_{bnd}}{\partial \rho} \) also proves useful. Using these substitutions and transforming equation (4.49) from a \( \rho \) grid to a \( \xi \) grid one obtains the magnetic field diffusion equation in terms of \( \iota \)

\[
\xi \frac{\partial \iota}{\partial \xi} - \xi^2 \kappa \frac{\partial \iota}{\partial \xi} = \frac{\partial}{\partial \xi} \left( \frac{c \eta_\parallel}{\mu} \left( \frac{dV}{d\xi} \right)^{-1} (RB_\phi) \frac{\partial}{\partial \xi} \left[ dV \frac{\partial \psi}{d\xi} \frac{|\nabla \xi|^2}{R^2} \right] \right) + \\
- \frac{\pi c}{\Phi_{lim}} \frac{\partial}{\partial \xi} \left[ \frac{\eta_\parallel \langle J \cdot B \rangle_{NI}}{(RB_\phi) < R^{-2}>} \right] 
\]
4.6 Magnetic field diffusion in TRANSPP

TRANSPP solves this equation for \( \iota \). Having obtained \( \iota \) one may find \( \langle J \cdot B \rangle_v \) and the loop voltage. The list of equations (4.54-4.60) show how other flux surfaced averaged quantities may be calculated on the \( \xi \) grid. \( L \) is the poloidal circumference (eqn. 4.55) and \( P_{\text{ohmic}} \) is the ohmic heating power (eqn. 4.59). In eqn. 4.56 the \( \frac{\partial A}{\partial \xi} \) is a cross-sectional area element and \( \langle J_\phi \rangle_A \) area integrates to the total toroidal plasma current.

\[
\psi = \int \frac{t \xi \Phi_{\text{lim}} d\xi}{\pi} \quad (4.54)
\]

\[
\langle B_{\text{pol}} \rangle_v = 2t \xi \Phi_{\text{lim}} \frac{\partial V^{-1}}{\partial \xi} L \quad (4.55)
\]

\[
\langle J_\phi \rangle_A = \frac{\Phi_{\text{lim}}}{2 \pi^2 \mu} \left( \frac{\partial A}{\partial \xi} \right)^{-1} \frac{\partial}{\partial \xi} \left[ \frac{\partial V}{\partial \xi} (t \xi) \left\langle \frac{|\nabla \xi|^2}{R^2} \right\rangle_v \right] \quad (4.56)
\]

\[
\langle E_\phi \rangle_v = \frac{V_I}{2\pi} \left( \frac{1}{R} \right)_v \quad (4.57)
\]

\[
\langle B_{\text{pol}}^2 \rangle_{8\pi} = \frac{1}{8\pi} \left( \frac{t \xi \Phi_{\text{lim}}}{\pi} \right)^2 \left\langle \frac{|\nabla \xi|^2}{R^2} \right\rangle_v \quad (4.58)
\]

\[
\langle P_{\text{ohmic}} \rangle_v = \langle J \cdot E \rangle_v = \langle J_\phi \rangle \cdot \langle E_\phi \rangle_v \quad (4.59)
\]

\[
\langle \nabla \cdot (E \times B) \rangle_v = \left( \frac{\partial V}{\partial \xi} \right)^{-1} \frac{\partial}{\partial \xi} \left[ \frac{V_I \Phi_{\text{lim}}}{2 \pi^2} \left( \frac{\partial V}{\partial \xi} (t \xi) \left\langle \frac{|\nabla \xi|^2}{R^2} \right\rangle_v \right) \right] \quad (4.60)
\]

4.6.3 Magnetic field diffusion algorithm

In TRANSPP equation 4.53 is solved in the subroutine SOLVIB. Restated following the nomenclature of the source code the equation may be written

\[
\xi \frac{\partial \iota}{\partial t} - \xi^2 \kappa \frac{\partial \iota}{\partial \xi} = \frac{\partial}{\partial \xi} \left( Z A \frac{\partial}{\partial \xi} \left\{ t Z B \right\} \right) - ZGX \quad (4.61)
\]

Comparison with equation 4.53 reveals the values of the variables \( ZA, ZB \) and \( ZGX \). This partial differential equation is solved so that the \( \iota \) profile can be advanced from one time step to the next. While the iota-bar profile needs to be established \textit{a priori} at the start of the TRANSPP run, subsequent time points always have an ‘old’ profile from which to build.

In the subroutine SOLVIB0, the axial value of \( \iota \) is obtained explicitly. The rest of the profile is then found using a standard matrix inversion technique (see below). L’Hôpital’s rule states that if \( \lim_{\xi \to 0} f(\xi) \) and \( \lim_{\xi \to 0} g(\xi) \) are both zero or both \( \pm \infty \) and furthermore if \( \lim_{\xi \to 0} \frac{f(\xi)}{g(\xi)} \) is not also of indeter-
minimize form \( \frac{\partial}{\partial \xi} \) then

\[
\lim_{\xi \to 0} \frac{f(\xi)}{g(\xi)} = \lim_{\xi \to >0} \frac{f'(\xi)}{g'(\xi)}
\]

Setting \( f(\xi) \) and \( g(\xi) \) equal to the left and right hand side of equation 4.61 respectively, it is valid to apply L'Hôpital’s rule. In the limit \( \xi \to 0 \) it is clear that \( f'(\xi) \to \frac{\partial t_{\text{axis}}}{\partial t} \) and so one has

\[
\frac{\partial t_{\text{axis}}}{\partial t} \bigg|_{\xi \to 0} = \frac{\partial^2}{\partial \xi^2} \{ ZA \frac{\partial}{\partial \xi} \{ \tau \text{ old} ZB \} \} - \frac{\partial}{\partial \xi} ZGX
\]

The code simply multiplies the right hand side of equation 4.63 by the magnetic time step \( DTB \) and adds it to the axial value of \( t \) from the previous time step. The resulting \( t_{\text{axis}} \) is then used in the tridiagonal matrix inversion step (see below) in which the \( t \) profile is calculated.

In general, it is also required that \( t \) on the plasma boundary is consistent with the measured plasma current. From Ampère’s Law, and following the derivation used to reach eqn 4.31, the total toroidal plasma current can be written as follows

\[
I_\phi = \frac{1}{2\pi \mu} \left( \frac{\partial V}{\partial \xi} \right) \left( \frac{\partial \psi}{\partial \xi} \right) \left( \frac{\left| \nabla \xi \right|^2}{R^2} \right)_v
\]

Using equation 4.50 the value of \( t \) on the plasma boundary, \( t_{\text{bnd}} \), may be written

\[
t_{\text{bnd}} = \frac{2\pi^2 \mu I_\phi}{\xi \frac{\partial V}{\partial \xi} \left( \frac{\left| \nabla \xi \right|^2}{R^2} \right)_v}
\]

The quantity \( t_{\text{bnd}} \) is then used as a boundary condition in the next step.

A tridiagonal matrix is a square matrix in which only the subdiagonal, diagonal and superdiagonal elements may be non-zero. It occurs in the solution to this problem since only the adjacent grid points will be used in the evaluation of the time derivative of \( t \) at any particular grid point. The matrix equation to be solved at grid point \( j \), where \( j_{\text{axis}} \leq j \leq j_{\text{bnd}} \), has the following form

\[
ZS(j) + UA(j) t (j + 1) + UB(j) * t (j) + UC(j) * t (j - 1) = 0
\]

In this equation \( ZS \) is the contribution to the change in \( t \) from the driven currents (this term is calculated elsewhere in the code). The matrix coefficients
are calculated from the following set of equations

\[
UA(j) = -\partial t \left[ \frac{\xi_i^2}{2\partial \xi} + \frac{ZA(j) \ast ZB(j+1)}{\xi \partial \xi^2} \right] \quad (4.67)
\]

\[
UB(j) = 1 + \partial t \left[ \frac{ZB(j)}{\xi \partial \xi} \left( \frac{ZA(j)}{\partial \xi} + \frac{ZA(j-1)}{\partial \xi} \right) \right] \quad (4.68)
\]

\[
UC(j) = -\partial t \left[ -\frac{\xi_i^2}{2\partial \xi} + \frac{ZA(j-1) \ast ZB(j-1)}{\xi \partial \xi^2} \right] \quad (4.69)
\]

In order to solve matrix equation 4.66 it is necessary to recast the problem as follows

\[
t(j+1) = UY(j) \ast t(j) + UX(j) \quad (4.70)
\]

where the diagonal matrices \(UY\) and \(UX\) are given by

\[
UX(j) = -\frac{ZS(j+1) + UA(j+1)UX(j+1)}{UA(j+1)UY(j+1) + UB(j+1)} \quad (4.71)
\]

\[
UY(j) = \frac{-UC(j+1)}{UA(j+1)UY(j+1) + UB(j+1)} \quad (4.72)
\]

By setting the boundary conditions \(UX(j_{bnd}) = t_{bnd}\) and \(UY(j_{bnd}) = 0\), the \(UX\) and \(UY\) profiles can be calculated, while ensuring that the solution is consistent with the total plasma current (alternatively, consistency with the measured loop voltage can be stipulated). By employing eqn 4.70, TRANSP builds a complete profile of the \(t\) beginning with the axial value, \(t_{axis}\).

The \(t\) profile that results from the matrix inversion is then input in SOLVI0 and an updated \(t_{axis}\) calculated. The matrix solution is repeated until the change in \(t_{axis}\) between successive iterations falls below a specified value.
4.7 Magnetic field diffusion simulation

4.7.1 Numerical simulations using Mathematica

An analytical treatment of the magnetic field diffusion in cylindrical geometry was provided in section 2.4.2. Recall the poloidal magnetic field diffusion equation in cylindrical geometry

$$\frac{\partial B_\theta}{\partial t} = \lambda_m \left( \frac{\partial^2 B_\theta}{\partial r^2} + \left[ \frac{1}{r} + \frac{1}{\lambda_m} \frac{\partial \lambda_m}{\partial r} \right] \frac{\partial B_\theta}{\partial r} + \left[ \frac{1}{r \lambda_m} \frac{\partial \lambda_m}{\partial r} - \frac{1}{r^2} \right] B_\theta \right) \quad (4.73)$$

Finding solutions to equation 4.73 when $\frac{\partial \lambda_m}{\partial r} \neq 0$ is problematic (see section 4.7.2 and also references [75] and [76] for analytical treatment examples). However, the expression can be evaluated numerically for specified functional representations of $\lambda_m(r)$. The software package Mathematica uses interpolating functions to numerically approximate the solution within given initial and boundary conditions.

As an example set of initial conditions, the diffusivity and initial $B_\theta$ profiles are modelled by second and sixth order polynomials respectively (see figure 4.5). By applying Ampère’s Law it is apparent that the initial $B_\theta$ profile approximates a current hole (see section 5.2.2). Since $\eta \propto T_e^{-3/2}$ (see
4.7 Magnetic field diffusion simulation

Figure 4.6: $B_\theta$ profile at 0.01s (solid), 0.5s (long dash) and 3.0s (short dash) where the diffusivity profile $\lambda_m = r^2 + 0.03$

section 2.4.3), the diffusivity profile corresponds to a temperature profile which is peaked on axis. Boundary conditions are also required. Figure 4.6 shows the $B_\theta$ profile at three time points during its evolution when its value at the outer edge is fixed at 1 T and at the axis at 0 T.

As the magnetic field diffuses, it approaches a steady state profile, i.e $\frac{\partial B_\theta}{\partial t} \rightarrow 0$. After 3.0 seconds in the example above, the magnetic field has essentially reached steady state ($\frac{\partial B_\theta}{\partial t} \approx 10^{-6}$). It is evident that the simplest magnetic field profile, i.e. a profile of uniform gradient, is not the steady state profile when $\lambda_m$ is non-uniform. In figure 4.7 the steady-state profiles that result when $\lambda_m$ assumes various spatial dependencies is shown. From the example profiles studied the following statement can be made about the slope of the steady-state profile, $B_\theta^{steady}$: if $\lambda_m$ increases with $r$ then the slope of $B_\theta^{steady}$ is greater than unity near the axis and less than unity near the boundary; on the other hand if $\lambda_m$ is decreasing with $r$ then the slope is less than unity near the axis and increases towards the plasma boundary (note that the slope of $B_\theta^{steady}$ in the case of uniform $\lambda_m$ is equal to unity). Similar results have also been found in general studies of variable diffusion coefficients [35].

By examination of equation 2.69, it can be seen that, in the case of uniform diffusivity, the evolution of the magnetic field is described by the
Figure 4.7: $B_\theta$ steady state profile for the following diffusivity cases; uniform $\lambda_m$ (solid line), $\lambda_m = r$ (dashed red), $\lambda_m = r^2$ (dotted red), $\lambda_m = 1 - r$ (dashed blue), $\lambda_m = 1 - r^2$ (dotted blue)

following simplified expression

$$B_\theta(r, t) = B_\theta^{steady}(r) + \left( B_\theta(r, \tau_R) - B_\theta^{steady}(r) \right) e^{-\lambda_m r^2 (t - \tau_R)} , \quad t > \tau_R \quad (4.74)$$

where $\tau_R = \frac{a^2}{\lambda_m \alpha_1^2}$ and $\lambda_{eff}$ is the effective diffusivity which replaces the $\lambda_o$ term in the uniform case. The resistive (or diffusion) time $\tau_R$ is proportional to the square of the plasma radius $a$. At the resistive time scale, exponential terms due to $n > 1$ values of $\alpha_n$ can be considered to have a negligible effect on the evolution (since the roots of the Bessel function $J_1(\alpha_n r)$ is a growing series, e.g $\alpha_1 = 3.83171, \alpha_2 = 7.01559, \alpha_3 = 10.1735$). In the case of non-uniform diffusivity, the question arises whether the rate of evolution is governed by the local value of resistivity or by some global quantity, i.e. we wish to determine the relationship between $\lambda_{eff}$ and $\lambda_m(r)$.

The slope of the $B_\theta$ evolution, at each radial location, is governed by the effective diffusivity. This can be found simply by calculating $\log \left( B_\theta(r, t) - B_\theta^{steady} \right)$, for times greater than $\tau_R$, but before the steady state is reached. The slope of this line, which is found by employing a least squares fit in Mathematica, when divided by $\alpha_1^2$ yields $\lambda_{eff}$. Applying this method of determining the effective diffusivity in the case of uniform resistivity, it is
found that it varies slightly from the uniform value (by about 0.1% approximately). It is surmised that this variation is an artefact of the Mathematica numerical calculation.

![Figure 4.8: Typical H-mode temperature profile (solid) and corresponding diffusivity profile (short dash, red), the latter scaled by 0.03 in the figure. The initial $B_\theta$ profile used in the example below is also included (long dash, blue), with associated axis labels on the right.](image)

If the effective diffusivity is then determined for the case of non-uniform resistivity, it is found that it varies across the profile within a similarly narrow range, which is much less than the total variation in the local resistivity. Therefore, within a small error, the evolution of $B_\theta$ at each radial location may accurately be described by a single quantity (see figure 4.10). This leads to the conjecture that the effective diffusivity $\lambda_{\text{eff}}$ is a global quantity, supported by analytical and more extensive numerical studies in this thesis (see sections 4.7.2 and 4.7.3). $\lambda_{\text{eff}}$ varies with the degree of the polynomial and the coefficients used to describe $\lambda_m(r)$.

**Diffusion in a H-mode plasma**

In order to illustrate these findings consider the magnetic field diffusion in the case of realistic resistivity profiles. For example, in high-confinement mode (H-mode) plasmas the temperature is typically peaked on axis with a steep gradient at the plasma edge (due to an edge confinement barrier). A
temperature profile given by $T_e(r) = (1 - r)e^{-1.5r^2 + 1.5r^4}$ represents these features satisfactorily (see figure 4.8). Furthermore, consider an initial poloidal magnetic field profile that approximates a current hole plasma. The effective diffusivity $\lambda_{\text{eff}}$ profile, calculated as described above, is shown in relation to the input diffusivity in figure 4.9. It is clear that a single global quantity describes the evolution at each time point, and furthermore that $\lambda_{\text{eff}}$ equals the input diffusivity in the region of the mid-point. In figure 4.10 the evolution of $B_\theta$ at three radial locations is shown. It is evident that at all three locations using $\lambda_{\text{eff}}$ in equation 4.74 provides an accurate fit to the solution. On the other hand using the local value of the diffusivity $\lambda_m(r)$ does not reproduce this accuracy, except at the midpoint where the local and effective diffusivities differ by a relatively small amount. It should be pointed out that the effective and local diffusivities need not coincide near the midpoint (For example, when $\lambda_m(r) = \lambda_o r^2$ they coincide at a point closer to the axis).

**Effective heat diffusivity**

A constant effective diffusivity is also found in analytical studies of the heat transport equation. The following analysis is taken from reference [75]. The
4.7 Magnetic field diffusion simulation

Figure 4.10: Evolution of $B_\theta$, starting at $t = \tau_R$, at radial locations 0.05m (top), 0.5m (middle) and 0.95m (bottom) as found by numerical simulation (black curve) and from equation 4.74 using the local diffusivity $\lambda_m(r)$ (short blue dash, triangle down) or the effective diffusivity $\lambda_{\text{eff}}$ (long red dash, triangle up). The input $\lambda_m(r)$ profile is typical of H-mode plasmas.

diffusion equation for the electron temperature $T_e$ in an infinite cylinder of radius $a$, and assuming any spatial variations only occur along the radius $r$, is given by

$$\frac{\partial T_e}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \kappa \frac{\partial T_e}{\partial r} \right) - b T_e + S(r,t) \quad (4.75)$$

where $\kappa = \frac{2}{3} \chi_e$, $\chi_e$ is the electron heat diffusivity, $b$ is the loss term and $S(r,t)$ is the source term. In this equation $\kappa$ is analogous to $\lambda_m$, and $T_e$ to $B_\theta$, in equation 4.73. If $\kappa$ is parameterised as follows

$$\kappa(r) = \kappa_o r^N, N \neq 2 \quad (4.76)$$
then the time dependent part of the Green’s function, which yields the solution for $T_e$, is given by

$$G_{\text{time part}} = \exp \left[ \left( -\kappa_0 \left(1 - \frac{N}{2}\right)^2 \beta_n a^N + b \right) (t - t') \right]$$

(4.77)

where $\beta_n$ are the roots of the Bessel function $J_{\nu}(\beta_n a)$. Note, the uniform diffusivity case is recovered by setting $N = 0$ above. Furthermore, it is clear that local values of the diffusivity $\kappa(r)$ do not feature in the time dependent part of the solution (see equation 9a in ref. [75]).

### 4.7.2 Analytic solutions in simple non-uniform cases.

**Monomial dependence**

Consider the case of monomial dependence, i.e. $\lambda_m(r)$ is parameterised as follows

$$\lambda_m(r) = \lambda_o r^N$$

(4.78)

where $N$ is an integer and $\lambda_o$ is a constant value. By substituting this into the magnetic field diffusion equation (equation 4.73) one obtains

$$\frac{\partial B_\theta}{\partial t} = \lambda_o r^N \frac{\partial^2 B_\theta}{\partial r^2} + \lambda_o (N + 1) r^{N-1} \frac{\partial B_\theta}{\partial r} + \lambda_o (N - 1) r^{N-2} B_\theta$$

(4.79)

The solution method used in section 2.4.2 is again employed. Define functions $u(r)$ and $v(t)$ and include a steady state term such that

$$B_\theta(r, t) = B_\theta^{\text{steady}}(r) + u(r) v(t)$$

(4.80)

The boundary and initial conditions are also restated

$$B_\theta = B_\theta^{\text{bnd}}, \, r = a, \, t \geq 0,$$

(4.81)

$$B_\theta = 0, \, r = 0, \, t \geq 0,$$

(4.82)

$$B_\theta = f(r), \, 0 < r < a, \, t = 0$$

(4.83)
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Steady State solutions
First, determine \( B^\text{steady}_\theta(r) \) by solving equation 4.79 in steady state
\[
\frac{\partial^2 B_\theta}{\partial r^2} + (N + 1) \frac{1}{r} \frac{\partial B_\theta}{\partial r} + (N - 1) \frac{1}{r^2} B_\theta = 0 \quad (4.84)
\]
The solution to the steady state equation turns out to have the following form
\[
B^\text{steady}_\theta(r) = c_1 r^{1-N} + \frac{c_2}{r}, \quad N \neq 2 \quad (4.85)
\]
\[
B^\text{steady}_\theta(r) = c_1 \frac{\log(r)}{r} + \frac{c_2}{r}, \quad N = 2 \quad (4.86)
\]
Note that for the uniform \( \lambda_m \) case (i.e. \( N = 0 \)) the physically admissible steady state solution is linear with \( r \). Furthermore, for all \( N < 0 \), in order to avoid the singularity on axis, set \( c_2 = 0 \) and we have
\[
B^\text{steady}_\theta(r) = c_1 r^{1-N}, \quad N \leq 0 \quad (4.87)
\]
However, for \( N > 1 \) the singularity on axis cannot be avoided, apart from the trivial solution \( B^\text{steady}_\theta(r) = 0 \). Considering the stipulated boundary conditions (eqns. 4.81 and 4.82) for \( N \geq 1 \), there are no analytical solutions to the magnetic field diffusion equation in steady state.

Steady state \( B_\theta \) profiles calculated by Mathematica, following the method discussed in section 4.7.1, can be compared with the analytical results For the cases of \( N = -1 \) and \( N = -2 \) the absence of a singularity on the axis allows the comparison of the exact analytical solutions (given by equation 4.87) with Mathematica numerical calculations (see figure 4.11). Note that the steady state profiles calculated in this section differ from those shown in figure 4.7 since in the latter case a constant was added to the \( \lambda_m \) profile.

Solutions to the separated variables

Solutions for the separated variables are found by setting
\[
\frac{1}{v} \frac{dv}{dt} = -\alpha^2 \lambda_o \quad (4.88)
\]
\[
\frac{\lambda_o r^N}{u} \frac{d^2 u}{dr^2} + \frac{\lambda_o (N + 1) r^{N-1} u}{u} \frac{du}{dr} + \lambda_o (N - 1) r^{N-2} = -\alpha^2 \lambda_o \quad (4.89)
\]
4.7 Magnetic field diffusion simulation

Figure 4.11: Numerical calculation of the $B_\theta$ steady state profile for the case of $\lambda_m = \frac{\lambda_o}{r}$ (solid, black line) and $\lambda_m = \frac{1}{r^2}$ (dash, blue line) compared to the analytic solution in both case (triangles up and triangles down respectively).

The solution to equation 4.88 is given by

$$v = e^{-\alpha^2 \lambda_o t}$$

Thus the problem has been reduced to solving the following ordinary differential equation, found by rearranging eqn. 4.89

$$\frac{d^2 u}{dr^2} + \left( \frac{N + 1}{r} \frac{du}{dr} \right) + \left( \frac{N - 1}{r^2} + \frac{\alpha^2}{r^N} \right) u = 0$$

For $N \neq 2$ the solution of the above equation involves Bessel functions and a general solution, similar to that obtained for the heat diffusion equation case in [75], may exist. In this section, the solution for selected values of $N$ will be given.

**Solution in the case of** $N = 1$, i.e. $\lambda_m = \lambda_o r$

In the case of $N = 1$ (i.e. the magnetic diffusivity is a linear function of radial position), the ODE becomes

$$\frac{d^2 u}{dr^2} + \left( \frac{2}{r} \frac{du}{dr} \right) + \left( \frac{\alpha^2}{r} \right) u = 0$$
Its solution is a linear combination of Bessel functions of the first and second kind.

\[ u = C_1 \frac{J_1(2\alpha \sqrt{r})}{\alpha \sqrt{r}} + C_2 \frac{Y_1(2\alpha \sqrt{r})}{\alpha \sqrt{r}} \]  

(4.93)

where \( C_1 \) and \( C_2 \) are constants of integration and \( \nu \) denotes the order of the Bessel function. Since \( J_1(z) \to 0 \) and \( Y_1(z) \to \infty \) as \( z \to 0 \), to avoid the unphysical singularity at the axis set \( C_2 = 0 \).

The solution, using the form given in equation 4.80, can then be expressed in the following form

\[ B_{\theta}(r, t) = B_{\theta \text{ steady}}(r) + \sum_{n=1}^{\infty} \frac{A_n J_1(2\alpha_n \sqrt{r})}{\alpha_n \sqrt{r}} e^{-\lambda_n \alpha_n^2 t} \]  

(4.94)

where the \( 2\alpha_n \sqrt{\alpha} \) are the tabulated roots of \( J_1(z) \). In order to satisfy the initial condition (eqn. 4.83) the function \( f(r) \) is given by

\[ f(r) = B_{\theta \text{ steady}}(r) + \sum_{n=1}^{\infty} \frac{A_n J_1(2\alpha_n \sqrt{r})}{\alpha_n \sqrt{r}} \]  

(4.95)

In order to express the coefficients \( A_n \) in terms of known quantities (following the method used in section 2.4.2) the variable change \( r \to (\frac{r'}{2})^2 \) is made, both sides of equation 4.95 are multiplied by \( (r')^2 J_1(\alpha_n r') \) and integrated from 0 to \( 2\sqrt{\alpha} \).

\[ \sum_{n=1}^{\infty} \frac{A_n}{\alpha_n} = \int_0^{2\sqrt{\alpha}} (r')^2 f(r) J_1(\alpha_n r') dr' - \int_0^{2\sqrt{\alpha}} (r')^2 B_{\theta \text{ steady}}(r) J_1(\alpha_n r') dr' 
= \frac{1}{2} \int_0^{2\sqrt{\alpha}} J_1(\alpha_n r') J_1(\alpha_n r') r' dr' \]  

(4.96)

The equation for the \( B_{\theta} \) in steady state (equation 4.85) for \( N = 1 \) may be written in terms of \( r' \) as

\[ B_{\theta \text{ steady}}(r') = c_1 - \frac{c_2}{(r')^2}, \quad N = 1 \]  

(4.97)

It can be shown that

\[ \int_0^{2\sqrt{\alpha}} (c_1(r')^2 - c_2) J_1(\alpha_n r') dr' = \frac{1}{\alpha_n} \left( c_2 J_0(2\alpha_n \sqrt{\alpha}) + a^2 c_1 J_2(2\alpha_n \sqrt{\alpha}) - c_2 \right) \]  

(4.98)

Using the identity given in equation 2.72 the coefficients may be written in
the following form

$$
\sum_{n=1}^{\infty} A_n = \frac{\int_0^{2\sqrt{a}}(r')^2 f(r) J_1(\alpha_n r')dr' - \frac{1}{\alpha_n} (c_2 J_0(2\alpha_n \sqrt{a}) + a^2 c_1 J_2(2\alpha_n \sqrt{a}) - c_2)}{4a[J_2(2\alpha_n \sqrt{a})]^2}
$$

(4.99)

Substitution of this expression for the coefficients into equation 4.94 will then yield a solution to the magnetic field diffusion equation where \( \lambda_m = \lambda_o r \). For simplicity, consider the case where the initial magnetic field is zero everywhere except at the boundary (i.e. set \( f(r) = 0 \)). Note that the constants \( c_1 \) and \( c_2 \) cannot be determined analytically.

$$
B_\theta(r,t) = B_\theta^{steady}(r) - \sum_{n=1}^{\infty} \left( \frac{c_2 J_0(2\alpha_n \sqrt{a}) + a^2 c_1 J_2(2\alpha_n \sqrt{a}) - c_2}{4a\alpha_n[J_2(2\alpha_n \sqrt{a})]^2} \right) \frac{J_1(2\alpha_n \sqrt{a})e^{-\lambda_o a^2 t}}{\sqrt{r}}
$$

(4.100)

**Solution in the case of \( N = -1 \), i.e. \( \lambda_m = \frac{2}{r} \)**

In the case of \( N = -1 \) the ODE given in equation 4.91 becomes

$$
\frac{d^2 u}{dr^2} - \left( \frac{2}{r^2} + \alpha^2 r \right) u = 0
$$

(4.101)

Its solution is, as in the previous example, a linear combination of Bessel functions of the first and second kind.

$$
u = C_1 \sqrt{r}(\alpha/3)^{1/3} J_1(\alpha_3^{2/3} r) + 2(-1)^{1/6} C_2 \sqrt{r}(\alpha/3)^{1/3} Y_1(\alpha_3^{2/3} r)
$$

(4.102)

where \( C1 \) and \( C2 \) are constants of integration and \( \nu \) denotes the order of the Bessel function. Since \( J_1(z) \to 0.0 \) and \( Y_1(z) \to \infty \) as \( z \to 0 \), to avoid the unphysical singularity at the axis set \( C_2 = 0.0 \).

The solution, using the form given in equation 4.80, can then be expressed in the following form

$$
B_\theta(r,t) = B_\theta^{steady}(r) + \sum_{n=1}^{\infty} A_n(\alpha/3)^{1/3} \sqrt{r} J_1(\alpha_3^{2/3} r)e^{-\lambda_o a^2 t}
$$

(4.103)

where the series \( \alpha_3^{2/3} \) are the tabulated roots of \( J_1(z) \). In order to satisfy
the initial condition (eqn. 4.83) the function \( f(r) \) is given by

\[
f(r) = B_{\theta}^{steady}(r) + \sum_{n=1}^{\infty} A_n (\frac{\alpha}{3})^{1/3} \sqrt{r} J_1 (\frac{2}{3} \alpha r^{3/2})
\] (4.104)

The steady state solution for cases of \( N < 0 \) is given by equation 4.87. Considering the boundary conditions it is evident that in this case

\[
B_{\theta}^{steady}(r) = B_{\theta}^{bnd} \left( \frac{r}{a} \right)^2
\] (4.105)

As before, in order to express the coefficients \( A_n \) in terms of known quantities, the variable change \( r \rightarrow \left( \frac{3r}{2} \right)^{2/3} \) is made, both sides of equation 4.104 are multiplied by \( r' J_1 (\alpha n r') \) and integrated from 0 to \( a' = \frac{2}{3} a^{3/2} \).

\[
\sum_{n=1}^{\infty} A_n \left( \frac{2 \alpha n}{9} \right)^{1/3} = \int_0^{a'} (r')^{4/3} f(r) J_1 (\alpha n r') dr' - \frac{(\frac{3}{2})^{1/3} B_{\theta}^{bnd} \alpha}{a} \int_0^{a'} (r')^{8/3} J_1 (\alpha n r') dr' \int_0^{a'} J_1 (\alpha n r') J_1 (\alpha m r') r' dr'
\] (4.106)

It can be shown that

\[
\int_0^{a'} (r')^{8/3} J_1 (\alpha n r') dr' = \frac{3}{28} \alpha_n (a')^{14/3} p F_q \left[ \frac{3}{7}, \frac{2}{3}, \frac{10}{3}, -\frac{1}{4} \alpha_n^2 (a')^2 \right]
\] (4.107)

where \( p F_q \) is the hypergeometric function. Using the identity given in equation 2.72, and once again assuming that \( f(r) = 0 \), the coefficients may be written in the following form

\[
\sum_{n=1}^{\infty} A_n \left( \frac{\alpha_n}{3} \right)^{1/3} = -\frac{9}{28} (\frac{3}{2})^{2/3} (a')^{8/3} B_{\theta}^{bnd} \alpha_n p F_q \left[ \frac{3}{7}, \frac{2}{3}, \frac{10}{3}, -\frac{1}{4} \alpha_n^2 (a')^2 \right] a \left[ J_2 (2 \alpha_n \sqrt{a}) \right]^2
\] (4.108)

Substituting this into equation 4.103, one obtains the solution to the magnetic field diffusion equation in the case of \( \lambda_m = \frac{\lambda}{r} \) and \( f(r) = 0 \).

\[
\frac{B_{\theta}(r,t)}{B_{\theta}^{bnd}} = \left( \frac{r}{a} \right)^2 + \sum_{n=1}^{\infty} -a^2 \alpha_n^2 \left[ \frac{3}{7} J_2 \left( 2 \alpha_n \sqrt{a} \right) \right]^2 \sqrt{r} J_1 (\frac{2}{3} \alpha r^{3/2}) e^{-\lambda_n \alpha_n^2 t}
\] (4.109)
Modified monomial dependence

Consider a simple modification to the monomial case so that the diffusivity is non-zero on axis. $\lambda_m(r)$ is parameterised as follows

$$\lambda_m(r) = \lambda_o(1 + (gr)^N)$$  \hspace{1cm} (4.110)

where $\lambda_o$ is the diffusivity on axis and $g$ is a free parameter. This form for $\lambda_m(r)$ better approximates realistic diffusivity profiles since a zero diffusivity on axis implies an infinite temperature there (for finite $Z_{\text{eff}}$). By substituting equation 4.110 into the magnetic field diffusion equation (equation 4.73) one obtains

$$\frac{\partial B_\theta}{\partial t} = \lambda_m \left( \frac{\partial^2 B_\theta}{\partial r^2} + \left( \frac{1}{r} + \frac{N(gr)^N}{r(1 + (gr)^N)} \right) \frac{\partial B_\theta}{\partial r} + \left( \frac{N(gr)^N}{r^2(1 + (gr)^N)} - \frac{1}{r^2} \right) B_\theta \right)$$  \hspace{1cm} (4.111)

Note, by setting $N = 0$ one recovers the magnetic field diffusion equation in the case of uniform diffusivity (eqn. 2.58). The solution form utilised in monomial and uniform case is again employed, that is

$$B_\theta(r,t) = B_{\theta \text{steady}}(r) + u(r).v(t)$$  \hspace{1cm} (4.112)

and the same boundary and initial conditions (equations 4.81 to 4.83) are assumed. Again we proceed by first finding solutions for the steady state case

$$\frac{\partial^2 B_\theta}{\partial r^2} + \left( \frac{1}{r} + \frac{N(gr)^N}{r(1 + (gr)^N)} \right) \frac{\partial B_\theta}{\partial r} + \left( \frac{N(gr)^N}{r^2(1 + (gr)^N)} - \frac{1}{r^2} \right) B_\theta = 0$$  \hspace{1cm} (4.113)

**Solution in the case of $N = 2$, i.e. $\lambda_m = \lambda_o (1 + g^2r^2)$**

It is unclear whether a general solution for all values of $N$ exists for the steady state equation. However, solutions can be found using Mathematica for selected values of $N$. In the case of $N = 2$ the solution has the following form

$$B_{\theta \text{steady}}(r) = \frac{c_1}{r} + c_2 \log (1 + g^2r^2), \quad N = 2$$  \hspace{1cm} (4.114)

To prevent an unphysical solution at the axis set $c_1 = 0$. The constant $c_2$ is established from the boundary condition at $r = a$ given by equation 4.81

$$B_{\theta \text{bnd}} = c_2 \log (1 + g^2a^2)$$  \hspace{1cm} (4.115)
and so the steady state solution is given by

$$B_\theta^{\text{steady}}(r) = \frac{aB_\theta^{\text{bnd}} \log (1 + g^2 r^2)}{r \log (1 + g^2 a^2)}, \quad N = 2 \quad (4.116)$$

Note that as $g \to 0$ above (i.e. $\lambda_m = \lambda_o$) the steady state profile in the case of uniform diffusivity is recovered (equation 2.63). Unlike the purely monomial case where the steady state profile depends only on the degree of the monomial $N$, the steady state solution in the modified case is also dependent on the value of $g$. However, it is independent of the choice of $\lambda_o$. Note that close to the axis, equation 4.116 is approximated by the following relation

$$B_\theta^{\text{steady}}(r) \approx \frac{aB_\theta^{\text{bnd}} g^2 r}{\log (1 + g^2 a^2)} \quad (4.117)$$

and so it is clear that $B_\theta \to 0$ as $r \to 0$ as required. The steady state profile calculated using equation 4.116 is compared with numerical calculations for $g = 2$ and $g = 20$ in figure 4.12.

Figure 4.12: Numerical calculation of the $B_\theta$ steady state profile for the case of $\lambda_m = \lambda_o(1 + (gr)^2)$ where $g = 2$ (solid, black line) and $g = 20$ (dash, blue line) compared to the analytic solution in both case (triangles up and triangles down respectively).
As before, the solutions for the separated variables are found by setting
\[
\frac{1}{v} \frac{dv}{dt} = -\beta^2 \lambda_o
\] (4.118)
\[
\lambda_m \left( \frac{1}{u} \frac{d^2 u}{dr^2} + \left( \frac{1}{r} + \frac{N(gr)^N}{r(1 + (gr)^N)} \right) \frac{1}{u} \frac{du}{dr} + \left( \frac{N(gr)^N}{r^2(1 + (gr)^N)} - \frac{1}{r^2} \right) \right) = -\beta^2 \lambda_o
\] (4.119)

The solution to equation 4.118 is given by
\[
v = e^{-\beta^2 \lambda_o t}
\] (4.120)

Equation 4.119 may be rewritten as follows
\[
\frac{d^2 u}{dr^2} + \left( \frac{1}{r} + \frac{N(gr)^N}{1 + (gr)^N} \right) \frac{du}{dr} + \left( \frac{N(gr)^N}{1 + (gr)^N} + \frac{\beta^2}{1 + (gr)^N} - \frac{1}{r^2} \right) u = 0
\] (4.121)

No solution for the case of general \(N \neq 0\) was found with the exception of \(N = 2\). In this case equation 4.121 becomes
\[
\frac{d^2 u}{dr^2} + \left( \frac{1}{r} + \frac{2g^2 r}{1 + g^2 r^2} \right) \frac{du}{dr} + \left( \frac{2g^2 + \beta^2}{1 + g^2 r^2} - \frac{1}{r^2} \right) u = 0
\] (4.122)

The solution to this equation is of the form\(^2\)
\[
u = cr \left( _2F_1 \left[ 1 - \frac{i\beta}{2g}, 1 + \frac{i\beta}{2g}, 2, -g^2 r^2 \right] \right)
\] (4.123)

where \(\iota = \sqrt{-1}\), \(_2F_1\) is Gauss’s hypergeometric function and \(c\) is an integration constant to be determined from the boundary conditions. In general, \(_2F_1[a, b; c; z]\) is given by the following hypergeometric series
\[
_2F_1[a, b; c; z] = 1 + \frac{ab}{1c} z + \frac{a(a + 1)b(b + 1)}{2c(c + 1)} z^2 + \ldots
\] (4.124)

This series may be written compactly using Pochhammer symbols [34]
\[
_2F_1[a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}
\] (4.125)

\(^2\)A second branch to the solution involving the Meijer G function is excluded since it is complex
where the Pochhammer symbol is defined by

\[(x)_n = \frac{\Gamma(x + n)}{\Gamma(x)}\]  

(4.126)

Although the hypergeometric function has complex arguments, one can see by inspecting the generating series that it is real valued for any choice of \(m\). In order to satisfy the boundary condition at the plasma edge, the solution to the magnetic field diffusion equation is written as follows

\[B_b(r, t) = -\frac{aB^{\text{bnd}}_b \log(1 + g^2r^2)}{r \log(1 + g^2a^2)} + \]
\[+ r \sum_{m=1}^{\infty} c_m 2F_1 \left[ -\frac{1}{2g}, 1 + \frac{t\beta_m}{2g}, 2, -g^2r^2 \right] e^{-\beta_m^2 \lambda_o t} \]

(4.127)

where the \(\beta_m\) terms are the roots of \(2F_1 \left[ -\frac{1}{2g}, 1 + \frac{t\beta}{2g}, 2, -g^2a^2 \right]\). Note that the location of the roots \(\beta_m\) depend both on the profile coefficient \(g\) and on the boundary position \(a\). In fact, as \(g \rightarrow 0\) (i.e. \(\lambda_m \approx \lambda_o\)) the roots approach the roots of the Bessel function of the first kind of order one, \(J_1(\alpha_n r)\) (which appear in the solution in the uniform case, see equation 2.76). In general, \(\beta_m\) increases with \(g\), as is evident in figure 4.13.

In order to satisfy the initial condition (eqn. 4.83) we require

\[f(r) = -\frac{aB^{\text{bnd}}_b \log(1 + g^2r^2)}{r \log(1 + g^2a^2)} + r \sum_{m=1}^{\infty} c_m 2F_1 \left[ -\frac{1}{2g}, 1 + \frac{t\beta_m}{2g}, 2, -g^2r^2 \right] \]

(4.128)

Rearranging in terms of the unknown constants \(c_m\) and integrating over the plasma radius we have

\[\sum_{m=1}^{\infty} c_m = \int_0^a f(r) dr - \frac{aB^{\text{bnd}}_b}{\log(1 + g^2a^2)} \int_0^a \frac{\log(1 + g^2r^2)}{r} dr \]
\[\int_0^a 2F_1 \left[ -\frac{1}{2g}, 1 + \frac{t\beta}{2g}, 2, -g^2r^2 \right] r dr \]

(4.129)

It can be shown that

\[\int_0^a 2F_1 \left[ -\frac{1}{2g}, 1 + \frac{t\beta}{2g}, 2, -g^2r^2 \right] r dr = -\frac{2}{\beta_m^2} \left[ 2F_1 \left[ -\frac{t\beta}{2g}, \frac{t\beta}{2g}, 1, -g^2a^2 \right] - 1 \right] \]

(4.130)

and also

\[\int_0^a \log(1 + g^2r^2) \frac{1}{r} dr = -\frac{1}{2} \text{Li}_2 \left[ -g^2a^2 \right] \]

(4.131)
Figure 4.13: Variation of the position of the first root in $\text{hypergeom} \left[ 1 - \frac{i\beta}{2g}, 1 + \frac{i\beta}{2g}, 2, -g^2a^2 \right]$ with the profile coefficient $g$ where $a = 1$. As indicated, the first root at $g = 0$ coincides with the first root in $J_1(\alpha_n a)$.

where $Li$ is the polylogarithm, also known as Jonquière’s function

$$Li_n [z] = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

(4.132)

and the full solution in the case of $\lambda_m = \lambda_o (1 + g^2 r^2)$ is given by

$$B_o(r, t) = \frac{aB_g^{\text{bnd}} \log (1 + g^2 r^2)}{r \log (1 + g^2 a^2)} - \frac{r}{2} \left( \int_0^a f(r) dr + \frac{aB_g^{\text{bnd}} Li_2 [-g^2a^2]}{2 \log (1 + g^2 a^2)} \right) \times$$

$$\times \sum_{m=1}^{\infty} \beta_m^2 \text{hypergeom} \left[ 1 - \frac{i\beta_m}{2g}, 1 + \frac{i\beta_m}{2g}, 2, -g^2r^2 \right] e^{-\beta_m^2 \lambda_o t}$$

(4.133)

It is of interest to apply the prescription for finding the effective diffusivity given in section 4.7.1 to this solution. Following equation 4.74 one may write

$$\lambda_{\text{eff}} = \frac{1}{\alpha^2} \frac{\left| \log (B_o(r, t) - B_o^{\text{steady}}) \right|}{|t - t^{\text{steady}}|}$$

(4.134)

Note that since $\lambda_{\text{eff}}$ is calculated for $t > \tau_R$, the higher order terms in $\beta_m$ can
be neglected. By applying this formula to equation 4.133 one obtains

\[ \lambda_{\text{eff}} = \frac{\lambda_o \beta_1^2}{\alpha_1^2} \]  

(4.135)

Note, that the resistive time \( \tau_R \) given by

\[ \tau_R = \frac{a^2}{\beta_1^2 \lambda_o} \]  

(4.136)

depends on the parameter \( g \) due to the dependence of \( \beta_1 \) on that quantity already discussed.

**Justification for a constant effective diffusivity**

In each of the solutions derived in this section (eqns. 4.100, 4.109 and 4.133), the time characteristic of the magnetic field evolution is independent of radial position, despite of the non-uniform \( \lambda_m \) profiles treated. A general statement in this respect can be made about the evolution of the magnetic field as given by the general magnetic field diffusion equation in cylindrical geometry (equation 4.73). Again assume the usual solution form, given by equation 4.112, to obtain (see equation 2.57)

\[
\frac{1}{v} \frac{dv}{dt} = \lambda_m \left( \frac{1}{u} \frac{d^2 u}{dr^2} + \left[ \frac{1}{r} + \frac{1}{\lambda_m} \frac{\partial \lambda_m}{\partial r} \right] \frac{1}{u} \frac{du}{dr} + \left[ \frac{1}{r \lambda_m} \frac{\partial \lambda_m}{\partial r} - \frac{1}{r^2} \right] \right) 
\]  

(4.137)

As before, this equation may be separated as follows

\[
\frac{1}{v} \frac{dv}{dt} = -\zeta^2 \lambda_o 
\]  

(4.138)

\[
\lambda_m \left( \frac{1}{u} \frac{d^2 u}{dr^2} + \left[ \frac{1}{r} + \frac{1}{\lambda_m} \frac{\partial \lambda_m}{\partial r} \right] \frac{1}{u} \frac{du}{dr} + \left[ \frac{1}{r \lambda_m} \frac{\partial \lambda_m}{\partial r} - \frac{1}{r^2} \right] \right) = -\zeta^2 \lambda_o 
\]  

(4.139)

where \( \zeta \) is a constant. The solution to the temporal part is given by

\[ v = v_o e^{-\zeta^2 \lambda_o t} \]  

(4.140)

where \( \lambda_o \) is given by the value of \( \lambda_m(r) \) at \( r = 0 \). Thus, the solution to the magnetic field diffusion equation for any non-uniform diffusivity profile
\( \lambda_m(r) \), where one exists, will always have the form

\[
B_\theta(r, t) = B_\theta^{\text{steady}}(r) + \sum_{n=1}^{\infty} A_n u_n(r) e^{-\lambda_n \zeta_n^2 t}
\]  
(4.141)

where \( A_n \) are integration constants and \( u_n(r) \) is the solution to equation 4.139 when the right hand side is \(-\zeta_n^2 \lambda_m\). Note that in the case of uniform \( \lambda_m \), \( u_n(r) \rightarrow J_1(\alpha_n r) \) and \( \zeta_n \rightarrow \alpha_n \) where \( \alpha_n \) are the roots of the Bessel function of the first kind. For sufficiently long times where all but the \( n = 1 \) terms in the above sum can be neglected, equation 4.141 simplifies to

\[
B_\theta(r, t) = B_\theta^{\text{steady}}(r) + A_1 u_1(r) e^{-\lambda_1 \zeta_1^2 t}, \quad t > \tau_R
\]  
(4.142)

From this equation it can be seen that the diffusion time is given by \( \tau_R = \frac{a^2}{\lambda_m \zeta_1^2} \) and the effective diffusivity, as defined by equation 4.74, is given by \( \lambda_{\text{eff}} = \frac{\lambda_m \zeta_1^2}{\alpha_1^2} \). Thus, it is clear that the effective diffusivity, and the diffusion time, are spatial constants for any diffusivity profile \( \lambda_m(r) \). Determining the value of the effective diffusivity for a given \( \lambda_m(r) \) profile will be the subject of the next section.

### 4.7.3 Model of effective diffusivity based on plasma parameters

It is desirable that an expression relating the effective diffusivity \( \lambda_{\text{eff}} \) to the input diffusivity \( \lambda_m(r) \) be obtained. Such an expression would allow the evolution of the magnetic field to be determined, without recourse to solving the magnetic field diffusion equation. To this end, in order that any expression be valid over a sufficiently wide parameter space, a Monte Carlo type analysis is undertaken. A sequence of electron temperature, density and \( Z_{\text{eff}} \) profiles are generated. Realistic profiles are obtained by selecting a suitable function for each quantity. The coefficients of these functions are then varied randomly within the limits of high-temperature tokamak plasmas. This set profiles immediately yield a set of resistivity (or magnetic diffusivity \( \lambda(r) \)) profiles. In each case an effective diffusivity \( \lambda_{\text{eff}} \) may be calculated using the method described above. Finally, using regression analysis, a relation between \( \lambda_{\text{eff}} \) and the generated data profiles may be obtained.

The electron temperature profiles, \( T_e \), are generated by the following func-
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\[ T_e(r) = \text{tedge} + (\text{amp} - \text{tedge})(1 + mr^2 + pr^4)(1 - r^n).\text{mod} \quad (4.143) \]

where the parameters \text{tedge}, \text{amp}, m, p, q and \text{mod} are defined as follows

- \text{amp} = 0.5 + 14R_1
- m = 2(R_2^2 - 0.5)
- p = 1 + R_3
- q = 1.5 + R_4
- \text{tedge} = \text{amp}(0.02 + .03R_5)
- \text{mod} = \int_r^{} \left( 1 + 0.2\Sigma_{n=2}^{5} R_{n+4} \cos(2n\pi r) \right) dr \quad (4.144)

and where \( R_1 - R_5 \) are uniformly distributed random numbers in the interval \([0, 1]\). The parameter \text{mod}, is essentially a Fourier modulation of the temperature profile. The temperature on axis generated by this parameterisation varies in the range \(1.0 - 20.0 \text{keV} \) approximately. The temperature profiles are peaked on axis and, depending on the parameters chosen, a steep gradient can exist near the plasma boundary, consistent with a H-mode plasma.

The simulated electron density profiles, \( n_e \), are given by the following function

\[ n_e(r) = \text{amp} \left( \text{Tanh} \left[ (1000 + c)(1 - r)^2 + n_{\text{edge}} \right] e^{-(ar^2 + br^4)} \right) \quad (4.145) \]

where the parameters \( n_{\text{edge}}, \text{amp}, a, b \) and \( c \) are defined as follows

- \text{amp} = (1.0 + 15R_1)10^{19}
- a = 0.05R_2
- b = 0.05R_3
- c = 100R_4
- \text{nedge} = 0.8R_5 \quad (4.146)

where, as before, \( R_1 - R_5 \) are uniformly distributed random numbers in the interval \([0, 1]\). In order to simulate an electron density pedestal, the Tanh function is employed to produce a sharp change in \( n_e \) near the plasma edge. The electron density on axis is in the range \( 1.0 \times 10^{19} - 1.5 \times 10^{20} \text{m}^{-3} \).
Finally, $Z_{\text{eff}}$ is parameterised in the following way

$$Z_{\text{eff}}(r) = Z_{\text{axis}} + (amp - Z_{\text{axis}})(ar^2 + br^4) \quad (4.147)$$

and the parameters $Z_{\text{axis}}, amp, a$ and $b$ are defined as follows

$$Z_{\text{axis}} = 1.0 + R_1$$
$$amp = Z_{\text{axis}} + 3R_2$$
$$a = 0.7R_3$$
$$b = 0.7R_4$$

This parameterisation ensures that $Z_{\text{eff}}$ rises monotonically towards the plasma edge, and its magnitude at the edge is in the range 1.0 – 5.0. An example set of $T_e, n_e, Z_{\text{eff}}$ profiles is shown in figure 4.14.

A series of 100 profiles are generated and in each case the effective diffusivity $\lambda_{\text{eff}}$ is determined. In every instance, the variation in effective diffusivity across the plasma is less than 1%; the global quantity $\lambda_{\text{eff}}$ is calculated by taking the line-average of this profile. The set of calculated $\lambda_{\text{eff}}$ values were
in the range $0.001 - 0.12 \, m^2 s^{-1}$. In order to ascertain how the effective diffusivity depends on the input data, consider the equation for Spitzer resistivity (eqn. 2.82) rewritten as follows

$$\eta_{\text{Spitzer}} = \gamma Z_{\text{eff}} T_e^{-3/2}$$

where by observation one can see $\gamma = 0.51 \frac{N(Z_m)^{1/2} \ln \Lambda}{e \sqrt{2\pi}}$. Note the Coulomb logarithm $\ln \Lambda$ depends on all three input profiles $T_e, n_e, Z_{\text{eff}}$. Likewise, the magnetic diffusivity equation equation may be written

$$\lambda_m(r) = \frac{\gamma}{\mu_0} Z_{\text{eff}} T_e^{-3/2}$$

where $\lambda_{\text{eff}}$ is plotted against the line-averaged temperature $\langle T_e \rangle$ and effective temperature $\langle T_{e*} \rangle$.

Figure 4.15: Effective diffusivity $\lambda_{\text{eff}}$ plotted against the line-averaged temperature $\langle T_e \rangle$ and effective temperature $\langle T_{e*} \rangle$.

It is postulated that the effective diffusivity is related to the mean of the input profiles, i.e. to some averaging of $\lambda_m(r)$. The following relation is proposed

$$\lambda_{\text{eff}} = A \frac{\langle \gamma \rangle}{\mu_0} \langle Z_{\text{eff}} \rangle \langle T_e \rangle^B$$

where A and B are quantities to be determined by regression analysis and $\langle f \rangle$ denotes the line-average of $f$. An improvement is made to the model by combining the $Z_{\text{eff}}$ and $T_e$ terms into an effective temperature $T_{e*} = T_e Z_{\text{eff}}^{-2/3}$.

$$\lambda_{\text{eff}} = A \frac{\langle \gamma \rangle}{\mu_0} \langle T_{e*} \rangle^B$$

In figure 4.15 the variation in $\lambda_{\text{eff}}$ with both $\langle T_e \rangle$ and $\langle T_{e*} \rangle$ is shown. It is
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Figure 4.16: Effective diffusivity $\lambda_{\text{eff}}$ predicted by model given by equation 4.152 plotted against calculated values on a Log-Log scale.

clear that there is much less scatter in the latter case, the variation in $\gamma$ being much less than in $Z_{\text{eff}}$. The NonlinearRegress function in Mathematica finds a least-squares fit to the $\lambda_{\text{eff}}$ data by varying the model parameters $A$ and $B$. The regression analysis yields $A = 1.232$ and $B = -1.5004$, and thus we have the relation $\lambda_{\text{eff}} \propto \langle T_e^* \rangle^{-1.5}$. Performing the regression analysis again, this time setting $B = -1.5$, leaving $A$ as the only model parameter, produces the following result

$$\lambda_{\text{eff}} = \frac{1.23}{\mu_0} \frac{\langle \gamma \rangle}{\langle T_e^* \rangle^{-3/2}}$$

(4.152)

The Mathematica output allows the accuracy of the model to be determined. The coefficient of determination $R^2$, i.e. the ratio of the error explained by the variance in the model predictors to the total error, is 0.9993. A quantity which may be thought of as the error of the regression analysis is the ratio of the root mean squared (rms) error in the prediction of the
effective diffusivity to the the root mean square of the effective diffusivity itself (i.e. $\frac{\text{rms}(\Delta \lambda)}{\text{rms}(\lambda)}$). Expressed as a percentage, this error is 2.7%. Figure 4.16 compares the model predictions of $\lambda_{\text{eff}}$ with the calculated values. It is appropriate to use a Log-Log scale, since a distribution uniform in $T_e$ will discriminate in favour of low $\lambda_{\text{eff}}$.

If profiles with $\langle T_e^* \rangle < 1\text{keV}$ are excluded from consideration in the regression analysis, the accuracy of the model is improved ($\frac{\text{rms}(\Delta \lambda)}{\text{rms}(\lambda)} = 1.1\%$). The decrease in accuracy at low effective temperature may be due to errors in the numerical solution of the magnetic field diffusion equation when the diffusion takes place over a relatively short time scale.

The effective diffusivity model may also be tested in the case of neoclassical resistivity. For example, the neoclassical formulation due to Sauter [49] (see also the corrigendum [50]) is given by

$$\eta_{\text{Sauter}} = \frac{1}{F_{33}} \eta_{\text{Spitzer}}$$

(4.153)

where $F_{33}$ is a function of the trapped particle fraction $f_t$, the electron collisionality $\nu_e$ and $Z_{\text{eff}}$. Given that the $F_{33}$ term may be subsumed into the effective temperature, as the $Z_{\text{eff}}$ profile was previously, it follows that the effective diffusivity model (equation 4.152) should also apply in this case (a caveat should be added to this conclusion: the validity of the effective diffusivity model has not been proven for all possible effective temperature profiles). Initial studies suggest the model does apply in the neoclassical case, though an exhaustive analysis has yet to be completed.
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Figure 4.17: Comparison of the ratios \( \frac{\langle \lambda_m(r) \rangle}{\lambda_{\text{eff}}} \) (black squares) and \( \frac{1}{\lambda_{\text{eff}} \langle \lambda_m(r) \rangle} \) (blue crosses) over a range of temperature profiles.

4.7.4 Alternative approaches to calculating the effective diffusivity

Diffusivity profile averaging

It is of interest to ascertain whether the effective diffusivity \( \lambda_{\text{eff}} \) can be obtained directly from the input diffusivity profile \( \lambda_m(r) \). The obvious starting point is to take the line-average of the diffusivity profile, \( \langle \lambda_m(r) \rangle \). An alternative approach, suggested by P. McCarthy [77], is to calculate the line-average of the inverse of the diffusivity profile \( \langle \frac{1}{\lambda_m(r)} \rangle \). Note that this is equivalent to averaging over the diffusion time as \( \tau \propto \frac{1}{\lambda_m} \). To evaluate the accuracy of these methods, the following ratios are calculated using a series of \( \lambda_m(r) \) profiles (using temperature profiles generated by equation 4.143 and assuming constant \( Z_{\text{eff}} \) and electron density profiles)

\[
\text{Ratio1} = \frac{\langle \lambda_m(r) \rangle}{\lambda_{\text{eff}}} \quad (4.154)
\]

\[
\text{Ratio2} = \frac{1}{\lambda_{\text{eff}} \langle \frac{1}{\lambda_m(r)} \rangle} \quad (4.155)
\]

The result of this comparison is shown in figure 4.17. It is clear that
the latter ratio, Ratio2, is closer to unity (its mean value is 0.79, compared
with a mean of 1.54 in the case of Ratio1) and has a smaller variance (1.1 ×
10^{-3} compared with 0.04). Finally, by performing a regression of the
values against the calculated \( \lambda_{\text{eff}} \) values, it is found that the following formula
provides an accurate prediction of \( \lambda_{\text{eff}} \) over the parameter regime studied

\[
\lambda_{\text{eff}} = \frac{1.27}{\langle \frac{1}{\lambda_m(r)} \rangle}
\] (4.156)

The model predictions are compared with the calculated values of \( \lambda_{\text{eff}} \) in
figure 4.18. The root mean square error ratio, \( \frac{\text{rms}(\Delta \lambda)}{\text{rms}(\lambda)} \), in this case is 2.9%.
The equivalent result obtained in the previous section, i.e. when evaluating
the accuracy of equation 4.152, was 2.7%. In fact, by comparing equation
4.156 with the equation 4.152, it is apparent that these are closely related
statements. Considering only the temperature component of the resistivity
it is clear that equation 4.156 may be written

\[
\frac{1}{\langle \frac{1}{\lambda_m(r)} \rangle} \sim \frac{1}{\langle \frac{1}{T_e^{-3/2}} \rangle} = \frac{1}{\langle T_e^{-3/2} \rangle}
\]

(4.157)

and the similar accuracy of these two models implies

\[
\langle T_e^{3/2} \rangle \approx \langle T_e \rangle^{3/2}
\]

(4.158)

Furthermore, the relative inaccuracy in using the mean of the diffusivity profile, \(\langle \lambda_m(r) \rangle\), as a predictor of \(\lambda_{\text{eff}}\) (as indicated by the high scatter of \(\frac{\lambda_m(r)}{\lambda_{\text{eff}}}\) values in figure 4.17) implies that the relative difference between \(\langle T_e^{-3/2} \rangle\) and \(\langle T_e^{-3/2} \rangle\) is much greater than between \(\langle T_e^{3/2} \rangle\) and \(\langle T_e^{3/2} \rangle\). This statement is supported by figure 4.19, in which the ratios of these quantities is compared. In the case of \(\langle \frac{T_e^{3/2}}{T_e^{3/2}} \rangle\), the mean over the datapoints in the figure is 1.06 and the variance of the ratio is \(4 \times 10^{-4}\). This observation is consistent with the similarity in the accuracy of the models given by equations 4.152 and 4.156.
Application of analytical results

Figure 4.20: Example $\lambda_m(r)$ profile (solid, black) and fitted $\lambda_{\text{mono}}$ profile (blue, dash). In this case $g = 3.85$ gives the best fit to the data.

Another approach is to apply the analytic results, obtained for simple non-uniform cases in section 4.7.2, to the $\lambda_m(r)$ profiles based on realistic plasma parameters obtained in the previous section. The following parameterisation of $\lambda_m(r)$ was investigated

$$\lambda_{\text{mono}}(r) = \lambda_o(1 + g^2 r^2)$$  \hspace{1cm} (4.159)

and it was found that $\beta_1$ is a function of $g$ (see figure 4.13). It was shown in section 4.7.2 that the effective diffusion time $\tau_R$ is given by $\frac{a^2}{\beta R \lambda_o}$ at times where the higher order terms in $\beta_m$ can be neglected (see equation 4.136). Thus, provided $\beta_1$ and $\lambda_o$ are known, the diffusion time follows. A function of the form $\lambda_{\text{mono}}$ which fits the realistic profile $\lambda_m(r)$ as closely as possible is calculated by a suitable choice of the parameters $\lambda_o$ and $g$ (see figure 4.20).

As shown in figure 4.21, $\beta_1$ is found by finding the first root of the function

$$2F_1 \left[1 - \frac{\vartheta}{2g}, 1 + \frac{\vartheta}{2g}, 2, -g^2 a^2\right].$$

$\lambda_o$ is assumed to be equal to the value of $\lambda_m(r)$ on axis. A prediction of the diffusion time $\tau_R^{\text{pred}}$ may now may be made. The magnetic field diffusion equation is then solved numerically, using the $\lambda_m(r)$ profile as input. As before, by finding the inverse of the slope of
4.7 Magnetic field diffusion simulation

Figure 4.21: Plot of the hypergeometric function \( {}_2F_1 \left[ 1 - \frac{i}{2g}, 1 + \frac{i}{2g}, 2, -g^2 a^2 \right] \) at the boundary in the case of \( g = 3.85 \). The location of the first root, \( \beta_1 \), is indicated.

\[
\log \left( B_\theta(r, t) - B_\theta^{\text{steady}}(r) \right),
\]
a numerical calculation of the diffusion time \( \tau_R^{\text{num}} \) is found. The accuracy of the predicted values can then be determined.

In figure 4.22 the diffusivity times \( \tau_R^{\text{pred}} \) with \( \tau_R^{\text{num}} \) are compared, here shown on a Log-Log scale, for a range of temperature profiles. The root mean square error ratio, \( \frac{\text{rms}(\Delta \tau)}{\text{rms}(\tau)} \), in this case is 11.4%. Thus, this method for determining the effective diffusion time is significantly less accurate than both the model drawn from the numerical analysis (given by equation 4.152) and the first model included in this section (given by equation 4.156). However, this model does have merit in that it is derived from analytical rather than numerical considerations.

### 4.7.5 Comparison with TRANSP output

It is of interest to ascertain whether the magnetic field diffusion equation as implemented in TRANSP will produce results consistent with those obtained using Mathematica. To this end, TRANSP is run using a suite of simulated data. No neutral beam or ICRF heating is included, the bootstrap model is switched off, and the simulated plasma geometry and total plasma current kept constant to simplify the analysis. For consistency with the straight-
4.7 Magnetic field diffusion simulation

Figure 4.22: Plot of $\tau^\text{pred}_R$ versus $\tau^\text{num}_R$, calculated using the set of temperature profiles generated by equation 4.143. $\tau^\text{pred}_D$ is calculated by fitting a monomial diffusivity profile $\lambda_\text{mono}$, for which the diffusion time is known analytically, to the realistic $\lambda_m(r)$ profile.

cylindrical geometry used by Mathematica, neoclassical effects are neglected in the TRANSP run. A pitch angle that approximates a current hole plasma is used to initiate the TRANSP run. The desired diffusivity profile is arrived at through the input of a suitably chosen temperature profile (a flat electron density and Zeff profile was assumed). For example, in the case of $\lambda_m \propto 1 - r^2$, the corresponding temperature profile is given by $T_e \propto (1 - 2r^2 + r^4)^{1/3}$ as input.

A factor which complicates the comparison is the difference in the way the boundary conditions are set in the two codes. In Mathematica, $B_\theta$ at the plasma edge and at the axis are specified as hard boundary conditions. By contrast, in TRANSP, $B_\theta$ may not be directly specified while the magnetic field is evolving (although the total plasma current may be specified).
Furthermore, it is $t$ rather than $B_\theta$ which is the evolved quantity. Due to Shafranov shifting the internal flux surface geometry (inside the fixed outer boundary) is not constant and this leads to a commensurate change in the edge temperature and, hence, resistivity and loop voltage at the plasma edge. By equation 4.65 it is clear that $t_{\text{bnd}}$, the value of $t$ at the edge, depends both on geometrical factors and the gradient of the loop voltage. Thus, it is apparent that fixing the plasma current $I_\phi$ does not ensure a fixed boundary condition in the TRANSP magnetic field diffusion calculation. In older versions of TRANSP, a model which took the plasma flux surfaces to be concentric toroids of circular cross section and in which Shafranov shifting was ignored, was available. This option has been removed in the latest versions.

**Steady state profiles**

In order to allow a comparison of the steady state $B_\theta$ profiles obtained by *Mathematica* with those calculated by TRANSP, the boundary value at the plasma edge in the case of *Mathematica* is set to that reached by TRANSP in steady state. This amounts to scaling the initial $B_\theta$ profile in *Mathematica* appropriately. Consider the case of $\lambda_m \propto r^2$; the steady state $B_\theta$ profile calculated by *Mathematica* in this case is shown in figure 4.7. The simulated
4.7 Magnetic field diffusion simulation

Figure 4.24: Steady state $B_\theta$ profiles calculated by Mathematica (solid, black) and TRANSP (dash, blue) in the case of $\lambda_m \propto r^2$. The initial $B_\theta$ profile used in Mathematica, scaled from that used in TRANSP, is also shown (dotted line).

electron temperature profile input to TRANSP, of form $T_e \propto r^{-\frac{4}{3}}$, and the corresponding diffusivity profile calculated by the code are shown in figure 4.23 (a flat electron density $n_e = 2.5 \times 10^{19} m^{-3}$ and a flat $Z_{\text{eff}} = 1.5$ are input). It proved necessary to modify $T_e$ near the axis to avoid numerical problems in TRANSP. In order to minimise Shafranov shifting the size of the plasma was set to $0.25 m$, much smaller than typical at JET. The steady state $B_\theta$ profiles calculated by both codes are shown in figure 4.24. In a similar fashion the steady state $B_\theta$ profiles in the case of $\lambda_m$ corresponding to typical H-mode plasma is calculated using TRANSP and Mathematica, the resulting profiles are shown in figure 4.25.

Effective diffusivity model in TRANSP

As explained above, it is difficult to compare the evolution of the magnetic field calculated by TRANSP and Mathematica directly. However, the validity of applying the effective diffusivity model (eqn. 4.152) to the TRANSP results may be investigated. 30 temperature profiles generated using the parameterisation given in section 4.7.3 are used in separate TRANSP runs. In each case, the effective diffusivity $\lambda_{\text{eff}}$ is calculated following the method
4.8 Conclusion

The transport analysis code TRANSP incorporates a comprehensive suite of physics modules and can input a wide range of data. The particle transport and energy balance models, and the diagnostic data relevant to this thesis, were described. The generalised magnetic field diffusion equation, and the algorithm used in its numerical solution, were described in detail. The $\xi$ grid, on which the equation is solved, allows for time dependent geometry, i.e.
plasmas whose flux surfaces change shape, move, or enclose varying amounts of toroidal flux. The solution of the generalised magnetic field diffusion equation also allows the calculation of other useful quantities such as the plasma current profile, voltage profile and poloidal field energy balance.

In addition, the poloidal magnetic field diffusion equation in cylindrical geometry was solved using the widely used software package Mathematica which, through the use of interpolating functions, is able to numerically solve partial differential equations when supplied with the appropriate initial and boundary conditions. It was assumed that $B_\theta$ varies along the radial coordinate only and its boundary values at $r = 0$ and $r = a$ are fixed in time. It was found, in the case of a spatially non-uniform diffusivity profile $\lambda_m(r)$, that the steady state $B_\theta$ profile is non-linear. Its shape was found to depend on the shape, as well as the absolute values, of $\lambda_m(r)$. It was further

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4_26.png}
\caption{Effective diffusivity $\lambda_{\text{eff}}$ predicted by model plotted against calculated values based on TRANSP output.}
\end{figure}
found, based on the Mathematica results, that the effective diffusivity with which the magnetic field evolves differs from the local input diffusivity $\lambda_m(r)$, and that its variation across the plasma radius varies little compared to the variation in $\lambda_m(r)$. In fact, in the cases examined, a single global quantity $\lambda_{\text{eff}}$ describes the evolution of the magnetic field to within an accuracy of $O(1\%)$.

An analytical approach to solving the magnetic field diffusion equation in cases of simple non-uniform diffusivity profiles was followed. Monomial, $\lambda_{\text{mono}}(r) = \lambda_o r^N$, and modified monomial, $\lambda_{\text{mono}}(r) = \lambda_o (1 + g^N r^N)$, cases were studied. In the latter case, a solution was found for $N = 2$, i.e. $\lambda_{\text{mono}}(r) = \lambda_o (1 + g^2 r^2)$. It was found that the effective diffusivity is a constant across the plasma and depends on the profile parameters $g$ and $\lambda_o$.

In order to establish the effective diffusivity $\lambda_{\text{eff}}$ for a given input diffusivity profile $\lambda_m(r)$, a Monte Carlo type analysis was completed. The $\lambda_{\text{eff}}$ values which results from a set of randomly spatially varied electron temperature, density and $Z_{\text{eff}}$ profiles were used as the basis for a regression analysis. It was found that $\lambda_{\text{eff}}$ is well described by an expression containing the line-average of the input profiles. If the $T_e$ and $Z_{\text{eff}}$ profiles are conflated into an effective temperature term $T_e^*$, a simple equation (eqn. 4.152) is obtained for $\lambda_{\text{eff}}$. Neglecting low temperature cases ($\langle T_e^* \rangle < 1\text{keV}$), the accuracy of this $\lambda_{\text{eff}}$ model is 1.1%. This result applies to a parameter space covering a broad range of high-temperature plasmas. Thus, the effective diffusivity model allows $\lambda_{\text{eff}}$, and hence the magnetic field evolution, to be determined without recourse to solving the magnetic field diffusion equation. The ability to calculate $\lambda_{\text{eff}}$ also allows the resistive time scale $\tau_R$ to be determined.

Two alternative approaches to finding the effective diffusivity were investigated. The first involves finding the average of the inverse of the input diffusivity $\langle \frac{1}{\lambda_m(r)} \rangle$. The model for $\lambda_{\text{eff}}$ based on this method has an accuracy comparable to the first model. It was found that the difference between the results of the two models is due to the method with which temperature profile is averaged in each case. Secondly, a model based on the analytical results was considered. This was based on fitting profiles of type $\lambda_{\text{mono}}(r) = \lambda_o (1 + g^2 r^2)$ to the realistic $\lambda_m(r)$ profiles. Since the diffusion time is known for a given set of $\lambda_o$ and $g$ values, this leads to an estimate of $\tau_R$ for the fitted profile.

---

3The concept of effective diffusivity has been studied in other fields (for example see [78]), though apparently not in the area of magnetic field diffusion.
This method of determining the effective diffusion time is less accurate than the other models. However, it is of interest to check the analytical solution against the numerical results.

Whether the TRANSP algorithm that solves the magnetic field diffusion equation is consistent with the Mathematica results was then investigated. The input data and settings used in TRANSP were arranged so that the conditions pertaining in the Mathematica analysis were replicated as closely as possible. However, differing implementations of the boundary conditions and the inclusion of Shafranov shifting of flux surfaces in the TRANSP code, were differences that could not be reconciled. Despite these differences, the steady state $B_\theta$ profile calculated by both codes were in quantitative agreement.

It was also found that, excepting a 2% increase in the constant factor, the effective diffusivity model derived from the Mathematica analysis also applies in the case of TRANSP (though with less accuracy). The TRANSP analysis covered a much narrower parameter space and comparisons are compromised by the aforementioned differences between the two codes. Despite these limitations, the TRANSP results are broadly consistent with the Mathematica calculations.

In comparing the TRANSP and Mathematica results directly, it has been shown that, in general, the results of the two codes are consistent. This consistency is, in effect, a verification of the algorithm used to solve the magnetic field diffusion equation in TRANSP. On a more fundamental level, a model has been developed that allows the effective diffusivity to be calculated. Hence, the resistive relaxation time of the current profile, which has been shown to be radially independent, can be determined requiring only the $Z_{\text{eff}}$ and electron temperature profiles.
Chapter 5

Comparison of TRANSP-evolved q-profiles with MSE constrained equilibrium fits on JET

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5.1 Introduction

The pitch angle predicted by the transport analysis code TRANSP was compared with motional Stark effect (MSE) data. TRANSP was initialised in early rampup phase using a measured pitch angle profile but evolved the current profile using the magnetic field diffusion equation thereafter. The comparison was carried out on a wide range of pulses, including cases where a current hole was established in the preheating phase. Beyond confirming that a neoclassical model of the resistivity is superior to the classical one in modelling the evolution of the current profile in JET, the results indicate that the TRANSP prediction is nearly always within the 95% confidence band of the MSE data. In addition, it is shown that the TRANSP q-profile is consistent with MHD data and Faraday rotation polarimetry data. The results presented here underpin the accuracy of the current profile predicted by a well-diagnosed TRANSP run.

The ability to model the current diffusion in tokamak plasmas using codes such as TRANSP [79], [80] is predicated on knowing the resistivity as well as the accurate modelling of non-inductive current drive. As well as the classical model due to Spitzer [36], a series of neoclassical models take into account the trapped particle fraction present in axisymmetric toroidal systems. This
5.1 Introduction

This paper is concerned *inter alia* with testing the validity of the classical and various neoclassical models on JET.

Studies on other tokamaks have confirmed that in large devices trapped particles must be taken into account [81]–[82]. The prediction of integrated quantities such as the surface voltage or mean Z-effective have been adopted as resistivity model tests at TFTR [62], JT60 [83] and JET [84] respectively. Other studies have inferred, for example from the existence of sawteeth [85], or during the pre-heating phase [86], that the resistivity is neoclassical at JET. A more thorough test is afforded by a direct comparison of the TRANSP magnetic pitch angle profile with motional Stark effect (MSE) measurements. Such a direct comparison has already found in favour of a neoclassical model of resistivity at TFTR [87] but, heretofore, has not been conducted at JET. Neoclassical resistivity was also inferred at DIII-D by calculating the electric field profile from a sequence of MSE constrained equilibria [88].

Further to confirming that a neoclassical model better describes the evolution of the plasma current profile at JET, it is desirable that an estimate of its accuracy be obtained. An accurate model of the resistivity would allow the inductive current to be calculated with confidence. Furthermore, a reliable bootstrap model and beam-driven current calculation would allow the total current profile to be evolved. A comprehensive Monte Carlo calculation ensures that the beam-driven current is accurately found [73], [89]. Used as an internal constraint in a magnetic equilibrium code, MSE pitch angle data allows the q-profile to be accurately reconstructed [60]. This safety factor profile (q-profile) can be compared with that obtained via current diffusion. The practical goal is that the current profile, with a known initial state e.g. from standard, reproducible current ramp-up scenarios, could be found with an acceptable accuracy even during times when MSE data is unavailable.

In this paper, we will compare the pitch angle profile measured by the MSE diagnostic with that predicted by the time dependent transport code TRANSP when using either Spitzer or various models of neoclassical resistivity and bootstrap current combined with other non-inductive contributions. Since the bootstrap current is a consequence of the trapped particles (and so is a neoclassical effect), it is not included when using Spitzer resistivity in the code.

The accuracy of the neoclassical predictions is determined by calculating the root mean squared error of the TRANSP prediction. In addition to con-
5.2 Analysis Methodology

The time dependent code TRANSP [80] in its 'primary' mode, uses the magnetic field diffusion equation

\[
\frac{\partial \mathbf{B}}{\partial t} = \frac{\eta}{\mu} \nabla^2 \mathbf{B} - \nabla \left( \frac{\eta}{\mu} \right) \times (\nabla \times \mathbf{B}) + \nabla \times (\eta J_{NI}) \quad (5.1)
\]

where \( \mathbf{B}, \eta \) and \( \mu \) are the magnetic field, electrical resistivity and magnetic permeability respectively, to advance the q-profile and hence the current profile in time. \( J_{NI} \) is the non-inductive current which is calculated independently in the code. For TRANSP to solve this partial differential equation numerically an initial magnetic geometry must be established and boundary conditions prescribed.

The method adopted by the present authors was to impose the magnetic pitch angle inferred from the MSE measurements as an initial condition. Using this initialisation the current profile was evolved via field diffusion. In this way, the TRANSP run was started with enough detailed information about the current profile to enable optimised shear and even current hole scenarios to be modelled. TRANSP was then run several times using different models of the resistivity \( \eta \) and bootstrap current but starting with the same initial conditions. The boundary condition imposed was that the TRANSP calculation of the total plasma current match the measured quantity. In this case, TRANSP predicts the surface voltage whose consistency with diagnostic measurements is partially dependent on the resistivity model chosen.

The parallel electrical resistivity in a plasma depends on the electron temperature and the local effective ion charge, \( Z_{\text{eff}} \) and in the case of neoclassical models on the trapped particle fraction. It also weakly depends on
the electron density through the Coulomb logarithm. An earlier neoclassical resistivity and bootstrap current model implemented in TRANSP, developed by Hirshmann [45], [46], is valid at low inverse aspect ratio, $\epsilon \ll 1$, and arbitrary electron collisionality $v_{e\ast}$. The multispecies code NCLASS [47], also included in TRANSP, is valid for arbitrary $\epsilon$ and $v_{e\ast}$. A third model, that of Sauter [49] (see also the corrigendum [50]) is a Z-effective parameterised neoclassical model which is as general as NCLASS but is more convenient to use. The domains of validity of the the various neoclassical models is discussed in more detail in ref. [82] and [49].

Where possible the electron cyclotron emission (ECE) temperature was used in the TRANSP calculation of the resistivity. A suprathermal electron population induced by lower hybrid heating, in some discharges, will corrupt the spectral shape of the ECE measured temperature in the outer regions of the plasma. In that case LIDAR Thomson scattering data is employed. It was found that LIDAR temperature data, appropriately smoothed, is largely consistent with ECE temperature data (It is acknowledged that systematic differences in the temperature profiles measured by these two diagnostics have been the subject of investigation at JET, however these differences did not significantly impact the results presented here). The profile measurement of $Z_{eff}$ from charge exchange was preferred to the line-integrated value deduced from visible bremsstrahlung. The plasma boundary, which needs to be supplied to TRANSP, was calculated by the interpretive equilibrium code EFIT [91] constrained by the external magnetics. Inside this EFIT-supplied boundary, a fixed-boundary equilibrium solver calculates the magnetic flux surface geometry using the kinetic pressure profile, including fast particle contributions, and the $q$-profile as constraints. TRANSP solves the poloidal magnetic diffusion equation within this fixed boundary without taking external magnetic measurements or internal MSE data into account.

The MSE diagnostic measures the polarisation angle of the Stark-split $D_\alpha$ emission at 25 locations (see figure 3.6). The measured angle $\gamma_m$ which includes contributions from all the field components needs to be transformed to $\gamma = \frac{B_z}{B_t}$ (where $B_z$ is the component of the poloidal magnetic field perpendicular to the major radius and $B_t$ is the toroidal field) along the midplane to be consistent with the TRANSP pitch angle using the following approximate relation [92].

$$\tan(\gamma_m) = \tan\gamma \frac{a_0}{a_5} + \frac{a_2}{a_5} \quad (5.2)$$
where \( a_0 - a_5 \) are geometry related factors. The error (i.e. one standard deviation) of the MSE measurement, largely due to calibration uncertainties, is about 0.2° [60]. In general, the MSE channels \#9 - \#19 lying in the radial range \( 3.03m < R < 3.50m \) are reliable (the outer channels being adversely affected by the limitations of the viewing geometry of the MSE diagnostic while the inner channels often suffer from low signal strength due to beam attenuation [60]). The typical radial location of the magnetic axis is 3.0m. Where a large toroidal velocity is measured by charge exchange spectroscopy a radial electric field correction to the MSE data was carried out [92].

A typical pulse involves Lower Hybrid Current Drive (LHCD) preheating followed by neutral beam and Ion Cyclotron Resonance Heating (ICRH) heating. Since the TRANSP run must start before the neutral beam phase (due to the Monte Carlo calculation that takes place during that phase) MSE data is unavailable initially. To circumvent this, the pitch angle used at the beginning of the TRANSP run was chosen to give agreement with experiment at the time of the first MSE profile. In cases where a current hole was established in the preheating phase the restriction that a negative current cannot be maintained in this region was adhered to [93]. In fact, the

\[ a_0 - a_5 \]

\[
\begin{align*}
\text{Time (s)} & \quad 2 & 4 & 6 & 8 \\
\text{Power (MW)} & \quad 2 & 4 & 6 & 8 & 10 & 12 \\
\text{Neutral Beam} & \quad \text{ICRH} & \quad \text{LHCD} \\
\text{Total Plasma Current (MA)} & \quad 0.5 & 1 & 1.5 & 2 & 2.5 \\
\text{start run} & \quad \text{start current diffusion} \\
\end{align*}
\]
TRANSP code will not allow a region of flat pitch angle around the axis, that is the signature feature of a current hole, to be input. This was replaced by a small but finite slope in pitch angle producing the closest approximation to a current hole that the code would allow ($q_{axis} < O(200)$). The run was initiated using this pitch angle to calculate a $q$-profile. To find the $q$-profile at time points after the pitch angle initialisation phase the magnetic field diffusion equation was employed. The predicted pitch angle profile was interpolated at the MSE channel locations to facilitate comparison with the data. This analysis, carried out for over 24 JET discharges, was repeated for each model of the resistivity and bootstrap current under investigation.

Since no LHCD model was included, an analysis of the formation of current holes and their subsequent maintenance (which at JET is mainly facilitated by off-axis LHCD) was beyond the remit of this study. However, the contraction of current holes following LHCD preheating was studied.

The set of discharges included in the analysis covers a range of injected neutral beam powers ($1 - 12$ MW), of ICRH input powers (up to 7 MW) and include cases both with and without LHCD preheating. Most of the discharges included were ITB (internal transport barrier) scenarios, due to the fact that accurate MSE data is frequently available in these cases. Plasma parameters typical of the discharges in the database include a flat-top plasma current of approximately 2 MA, beta poloidal in the range $0.2 - 0.5$, toroidal magnetic field in the range $2.5 - 3.25$ T and electron temperature on axis in the range $2 - 8$ keV. The beam-driven current was calculated in each shot using a Monte Carlo neutral beam package and including neoclassical effects [89]. The maximum percentage of bootstrap current and beam-driven current to total plasma current averaged over the database was 17% and 10% respectively. ICRH effects were modelled using a bounce-averaged Fokker-Planck code coupled with the full-wave deposition code SPRUCE [94]. In all cases the standard scenario of ICRH heating was followed, i.e. minority heating in a deuterium plasma. ICRH was not used as an additional source of current drive, simplifying the analysis. MSE data was usually present during the current ramp up phase, thus allowing the measurement of a rapidly changing pitch angle and a more exacting test of the resistivity and bootstrap models. The salient features in the timing of a typical TRANSP run are illustrated in relation to the non-ohmic power input in figure 5.1.
5.2 Analysis Methodology

5.2.1 Initial Pitch Angle

In order to obtain a reliable pitch angle to initiate the current diffusion in TRANSP the following procedure was adopted. To begin, the MSE data was transformed onto the major radius using equation 5.2. Then those time points where the data was deemed corrupted were discarded. For example, at the time of neutral beam switch-on transient effects are often seen in the MSE data. In order to dampen the effects of noise on the data, the average over the first 10 time points were taken (equivalent to smoothing over 10 ms). The pitch angle is then extrapolated backwards in time using the total plasma current to the start time of the TRANSP run (typically 0.2s before). Even though the current diffusivity varies over the major radius, scaling the pitch angle with the total plasma current over the short time scale involved proved to be a reasonable approximation.

In the case where the MSE data indicates the presence of a current hole, further modification of the pitch angle is required. The current hole region, defined by a constant pitch angle on either side of the magnetic axis, is often contaminated by noise. TRANSP requires that the pitch angle monotonically increase along the major radius from a negative value at the inboard boundary of the plasma, through zero uniquely at the magnetic axis, and becoming increasingly positive toward the outboard boundary. An approximation to the current hole was constructed by imposing a linear fit to the pitch angle data over the current hole region (see figure 5.2). The finite slope of this line was chosen to be as close to zero as possible and in accordance with the magnetic axis separately found by EFIT. The result is a current profile with a trough of low current around the axis - the closest approximation to a current hole allowable with the current version of TRANSP.

Having obtained an initial pitch angle the next step is to set up the TRANSP run to input it correctly. For a current hole case, the small but finite slope around the axis will result in a large value of safety factor on axis which could cause the code to crash. This will often happen in the poloidal field solver due to the time steps becoming increasingly small in order to restrict the variation of $q$ on axis. To avoid this, an upper limit on $q$ may be imposed. If the TRANSP solution points to a value of $q$ above this level, a non-physical current around the axis is applied to lower the safety factor in the locale.

Reliable MSE data is typically limited in radial extent to 3.55m. Since
5.2 Analysis Methodology

TRANSP requires the pitch angle over the entire outboard half of the plasma, the pitch angle beyond that point needs to be supplied. The code completes the pitch angle profile using the boundary constraint that the total plasma current be conserved and assuming that the plasma current is proportional to the electron temperature according to the relation $j \propto T^2$ in the intervening region.

The evolution of the plasma current for the first few time points of the TRANSP run was determined by the input pitch angle. During this time the poloidal field solver was inverted and a resistivity profile output. Noise in the MSE data may result in negative resistivity values (these are indicative of a need to smooth the data but will not cause TRANSP to crash). The $\tau$ profile is first established from the poloidal magnetic field found from the pitch angle data. Using eqn. 4.54 and noting that $|B_\theta| = \frac{1}{R} |\nabla \psi| = \frac{1}{R} \frac{\partial \psi}{\partial \rho} |\nabla \rho|$ one has the relation

$$\tau = \frac{\pi B_\theta R}{\Phi_{\text{bnd}} \xi d\xi}$$  \hspace{1cm} (5.3)

Ampere’s law applied to the $\tau$ profile yields the current profile (eqn. 4.56), while Faraday’s law allows the voltage gradient to be calculated (eqn. 4.28). Finally, the resistivity can be calculated using Ohm’s Law. At the end of this initialisation phase, a transition period allows a smooth transfer of control.
of the plasma current evolution back to the poloidal field diffusion solver.

If the TRANSP calculated pitch angle is close to the first MSE measurement then it can be said that the initial pitch angle used was a good approximation. However, it was found that the TRANSP calculated pitch angle is invariably more smooth than the measurements. Thus, in order to compare the subsequent evolution of the TRANSP pitch angle to the measurements from a common starting point, the data at each radial location was first shifted to match the initial predicted pitch angle.

5.2.2 EFIT and current holes

![Image of pitch angle and current density plots]

Figure 5.3: Fit to measured pitch angle (crosses) made by two EFIT runs, one of which is optimally weighted (dashed line) and the other over-constrained to match the MSE data (solid line), and the corresponding current profiles that result (top).

In addition to the pitch angle comparisons, current profiles which were produced by EFIT with MSE data as an internal constraint were compared to those calculated by TRANSP. Many of the pulses in the database (see below) had MSE pitch angle profiles which bear the hallmarks of a current hole. EFIT is unable to converge to an equilibrium solution in these cases. This is due to the fact that the basis functions that are used to represent the
current density profile in EFIT utilise normalised poloidal flux $\psi$. However, over a region of near-zero current density $\psi$ is almost constant and so using it to uniquely label the flux surfaces is problematic. A modified version of the EFIT code, produced by Nick Hawkes, allows the calculation of a reasonable current profile. In this version, the flux surface geometry is first found by EFIT using the magnetics data only to constrain the solution (eqn. 5.4).

$$\Delta^* \psi_{\text{mag}}^n = -\mu_0 R^2 p'(\psi_{\text{mag}}^n) - \mu^2 F(\psi_{\text{mag}}^n) F'(\psi_{\text{mag}}^n)$$ (5.4)

Starting with this magnetic surface geometry, $\psi_{\text{mag}}^n$, the coefficients in the current profile representation (eqn. 5.5) are recalculated with MSE pitch angle data input as an additional constraint. This is done in accordance with least squares minimisation of $\chi^2$ which allows a corrected solution $\psi_{\text{new}}$ to be found (eqn. 5.6).

$$J_\phi(\psi) = \sum_{k=1}^{N_p} C^p_k \hat{\psi}^{k-1} + \sum_{k=1}^{N_F} C^F_k \hat{\psi}^{k-1}$$ (5.5)

$$\Delta^* \psi_{\text{new}} = -\mu_0 R J_\phi(C_{\text{new}}, \psi_{\text{mag}}^n)$$ (5.6)

The resulting flux map $\psi_{\text{new}}$ has differently shaped flux surfaces than $\psi_{\text{mag}}^n$ and different values of $\psi$ on each surface. Nevertheless, Nick Hawkes’ results indicate reasonable agreement between the two solutions $\psi_{\text{new}}$ and $\psi_{\text{mag}}^n$, which suggests this method of determining the current profile is consistent with the Grad-Shafranov equation [92]. Following this method, EFIT can produce a current profile which is zero or even negative (actually positive by JET convention) on axis.

When including MSE data in the EFIT equilibrium calculation the number of degrees of freedom allowed in the current profile parameterisation was carefully selected. Too much freedom results in a wildly varying profile due to an over-emphasis on the MSE data. Spline knots were strategically placed in order to elucidate the details of the current profile around the axis. Figure 5.3 depicts an example in which an over-constrained fit to the MSE data can result in a negative current density around the axis.
5.2.3 The Database

A number of factors were considered when determining which JET pulses should be selected for analysis. The quality of the MSE data was the primary one. This included both the length of time for which good data was available and its radial extent. Data that coincided with the current ramp-up was preferred to ensure a more rapid change in pitch angle. The best quality data was obtained through selective firing of neutral beam PINIs during the pulse to avoid contamination of the MSE signal. For a more reliable TRANSP run, charge exchange data corrected using CHEAP analysis was desirable. In particular, a corrected $Z_{\text{eff}}$ profile ensured a more accurate resistivity calculation. Despite this, a line-averaged $Z_{\text{eff}}$ proved sufficient for a number of the pulses studied. The temperature measured by the ECE diagnostic was preferable but smoothed LIDAR sufficed in its stead.

A preponderance of MSE data exists for pulses with optimised shear. This is due both to an interest in the details of the current profile of these pulses and the difficulty in obtaining good data during high powered Elmy H-mode pulses. Due to their close association with optimised shear scenarios, many of pulses in the database were designed to study internal transport barriers (ITBs). The dependence of ITBs on plasma shaping, electron ITBs, long pulse ITBs and impurity transport in ITBs are some of the topics covered. Other areas investigated by the pulses in the database include toroidal Alfvén eigenmodes in optimised shear discharges, profile stiffness and current hole clamping. Many of the pulses analyzed showed signs of a current hole around the magnetic axis. A group of pulses which were designed to measure the radial electric field and current profile in electron ITBs provided some especially good quality data. In these pulses minimal neutral beam power was applied in a configuration that increased the accuracy of the MSE data and also allowed the toroidal rotation of the plasma to be measured so an electric field correction could later be made.
Figure 5.4: Comparison of the surface voltage calculated by TRANSP with that measured. Predictions were made using NCLASS (triangles) and Spitzer (squares) resistivity models. As well as the $y = x$ line (dotted), the linear fits (of form $y = \text{slope} \times x$) to the NCLASS (bold line) and Spitzer (dash) predictions are included. An error bar of 10% applies to the measured voltage; two are included in the figure. The error bar (one standard deviation) in the TRANSP prediction, calculated by varying the data within experimental uncertainties, is also shown.

5.3 Analysis of Results

5.3.1 Consistency of TRANSP predicted loop voltage with data

The surface voltage predicted by TRANSP is dependent on the resistivity and bootstrap model used. When this is compared to the surface voltage measured by diagnostics [95] it is clear that the NCLASS neoclassical model is more accurate than Spitzer (figure 5.4). The slope of the linear fit to the calculated voltage versus measured voltage data is 1.07 and 0.6 in neoclassical case and Spitzer case respectively. The surface voltage calculated by the Hirshmann neoclassical model is very similar to NCLASS. The scatter evident in the figure depends in a complicated manner on the uncertainty in the electron temperature and $Z_{\text{eff}}$, and since the plasma current profile is not in equilibrium, on the history of the evolution of the plasma current and the
5.3 Analysis of Results

Figure 5.5: Comparison of the pitch angle calculated by TRANSP with that measured (full line) by MSE at channel (a) 3.08m and (b) 3.36m. Predictions were made using NCLASS (short dash), Sauter (dash-dot, blue), Hirshmann (long dash, red) and Spitzer (dots) resistivity models. The rmse of the NCLASS and Spitzer model predictions were 0.18° and 0.46° respectively at channel (a) and 0.22° and 0.66° respectively at channel (b). The raw MSE data was averaged over ±15ms. The error bar (one standard deviation) in the MSE data and in the TRANSP predictions is also shown.

initial conditions prescribed [96]. Though the comparison of this integrated quantity finds strongly in favour of the neoclassical models, it cannot resolve possible variations along the plasma radius.

In order to establish the error in the TRANSP prediction due to the uncertainty in the experimental data employed, a series of 30 runs was carried out in which the electron density, temperature and $Z_{\text{eff}}$ data was varied within their experimental uncertainties. One standard deviation in the TRANSP calculation of voltage or pitch angle, and other quantities of interest, determines the associated error bars in figures 5.4-5.14.
5.3.2 Comparison of TRANSP output with pitch angle data

To begin a more detailed comparison the MSE pitch angle data was employed. JET discharge #53492 was preheated with LHCD and then powered with 8MW neutral beams while avoiding those Positive Ion Neutral Injectors (PINIs) which adversely affect the quality of the MSE data. ICRH power, which can also sometimes interfere with the MSE detectors, is absent in this discharge. Thus, the MSE data from this discharge suffered minimal interference from these heating sources. A narrow current hole established during the LHCD phase contracts during the main heating phase. The beam-driven and bootstrap currents accounted for, at their respective maxima, just 5% and 13.5% of the total plasma current. Using the method described in the previous section, a TRANSP predicted pitch angle profile was calculated for each resistivity model.

![Pitch Angle Profile](image)

Figure 5.6: Measured pitch angle profile (symbols) at the start(squares), at an intermediate time (stars) and at the end(circles) of the MSE data, and the TRANSP calculation using the NCLASS neoclassical model (solid)

The neoclassical prediction (for each neoclassical model) was much superior to Spitzer when compared with the MSE data measured at channels #9 – #19. Furthermore, for all 14 MSE channels in the range $2.75 \text{m} < R < 3.41 \text{m}$, the root mean squared error (rmse) of the NCLASS neoclassical
5.3 Analysis of Results

The prediction was less than 0.25° (i.e. comparable with the error associated with the data). The neoclassical predictions of the Hirshmann and Sauter models were similar to NCLASS. By contrast, the rmse of the Spitzer prediction, when averaged over the same set of channels was 0.47°. The accuracy of the TRANSP prediction of the pitch angle when using a neoclassical model, in contrast to the markedly more inaccurate prediction of the Spitzer model, is illustrated for two MSE channels plotted in figure 5.5. A comparison of pitch angle profiles at the beginning, at an intermediate time, and at the end of the MSE phase is shown in figure 5.6.

Figure 5.7: Current density and iota-bar profiles at the start (squares), at an intermediate time (stars) and at the end (circles) of MSE data calculated by EFIT+MSE (solid line) and TRANSP (dashes) (shot 53492 at times 4.1s, 5.1s and 7.0s)

The accuracy with which the pitch angle is evolved by TRANSP is reflected in the current profile evolution. The TRANSP iota-bar (iota-bar is the inverse of the safety factor $q$) and current density profile and that found by EFIT with MSE constraint is shown at the beginning, at an intermediate time, and at the end of the MSE phase in figure 5.7. Since TRANSP is initialised using pitch angle data, its current profile around the magnetic
axis also tends towards a current hole. However, zero current around the magnetic axis cannot be admitted by the code due to numerical instabilities at overly high values of q on axis. Furthermore, since the TRANSP run starts about 0.2s before the MSE phase it follows that the pitch angle it calculates at the start of that phase (via field diffusion) will differ from the measurements (as is evident in fig 5.7). This difference is due in part to uncertainties in the data used to calculate the resistivity. The figure shows that the TRANSP evolved current profile is consistent with the EFIT current profile. In particular, the contraction of the current hole is well predicted by TRANSP. The presence of a narrow off-axis peak in the EFIT current profile is predicted by LHCD deposition calculations [97]. It should be noted that the comparison of q profile differences from distinct codes is more difficult than the corresponding iota-bar profile comparison due to the strongly non-uniform (heteroscedastic) nature of the error propagation in q. For a given uncertainty in the MSE measurement $\Delta \gamma$, the uncertainty in q at the location of the MSE channel scales as $\Delta q \propto q^2 \Delta \gamma$, i.e. as the square of the q value. In contrast, the uncertainty in iota-bar at a fixed location is independent of the value of iota-bar [98].

From the database of pulses on which a similar analysis was carried out the conditions necessary for an accurate prediction of the pitch angle by TRANSP could be established. In the three comparisons shown in figure 5.8 differing behaviours of pitch angle evolution are well predicted by the code. The selected channels were those at which certain characteristics of the time evolution of the pitch angle, specific to each pulse, was most prominent. However it should be noted that when subsequently calculating the accuracy of the TRANSP prediction, all reliable MSE channels (as identified in section 2) were included. In figure 5.9 plasma parameters which are pertinent to the behaviour of the pitch angle in these three pulses are plotted. In the figure 5.8 (a) (pulse #52656 at 3.4m) the pitch angle at 5.8s rises sharply following a reduction in neutral beam and ICRH power and a consequent shifting inwards of the plasma axis (see figure 5.9 (a)). Conversely, in figure 5.8 (b) (pulse #52658 at 3.36m) the magnetic axis moves outwards in the range 3.8s to 4.6s as input power is increased. At 4.8s the competing process of pitch angle increasing due to inward current diffusion begins to dominate. In figure 5.8 (c) (pulse #53493 at 3.22m) the pitch angle data is available only intermittently due to the required neutral
Figure 5.8: Comparison of the pitch angle calculated by TRANSP with that measured (full line) by MSE at one channel from three pulses: (a) 52656 at 3.4m (b) 52658 at 3.36m and (c) 53493 at 3.22m. Predictions were made using NCLASS (dashes, red) and Spitzer (dash-dot, blue) resistivity models. Beta poloidal (dots) is also included.

beam PINI being fired in so called 'blips' (figure 5.8 (c)). In this case, the non-monotonicity of the pitch angle time trace is due to the ramp up and subsequent fall in the ICRH and ohmic powers. TRANSP accurately predicts the pitch angle at each 'blip' and therefore, one might conclude, at the intervening times. The evidence of Shafranov shifting in the pitch angle in all three cases is well predicted by TRANSP when using a neoclassical model for the resistivity and bootstrap current.

As a measure of the accuracy with which the pitch angle is predicted by TRANSP, the root mean squared error \( rmse_\gamma \) was calculated using all time points for each MSE channel and each pulse (and each model of resistivity
Figure 5.9: Total plasma current (dash, blue), neutral beam (triangles) and ICRH (stars, red) power corresponding to the three pulses in fig 5.8.

and bootstrap current under consideration) in turn - see equation 3.

\[
rmse_{\gamma} = \sqrt{\frac{\sum_{time}^{N_{time}} (\gamma_{DATA} - \gamma_{TRANSP})^2}{N_{time}}}
\]  

(5.7)

where \(\gamma_{DATA}\), \(\gamma_{TRANSP}\) are the measured and predicted pitch angles respectively and \(N_{time}\) is the number of time points summed over for a particular pulse. In figure 5.10 the rmse at channel 3.4m, against mean neutral beam and mean ICRF power, is plotted. In both cases no clear dependency on the level of input power is evident. A similar lack of dependency prevails at MSE channels #9 – #19.

An analysis of the database showed that the mean rmse of the TRANSP prediction (when using either the Hirshmann, NCLASS or Sauter neoclassical models) of the pitch angle at MSE channels #9 – #19 (identified in
5.3 Analysis of Results

Figure 5.10: Root mean squared error of pitch angle prediction against neutral beam power (squares) and ICRH power (stars) for each shot in the database at 3.4m

section 5.2) was approximately 0.35°. This is within two standard deviations ($2 \times 0.2°$) of the MSE pitch angle data. The variation in this figure due to the neoclassical model used was just 3% (the Hirshmann model having the highest rms). The mean rms calculated at the four MSE channels in the range 3.12m-3.27m were above this global mean with a maximum mean rms of 0.4° being recorded at 3.22m. By contrast, when the Spitzer model of resistivity was used the mean rms over all channels was 0.58°. If the NCLASS bootstrap current is artificially included in the TRANSP run while using Spitzer resistivity the mean rms was 0.53°. Thus 78% of the difference in accuracy between the Spitzer and neoclassical predictions of the pitch angle is due to the resistivity component of the calculation.

It is also of interest to compare the $\iota$ profile of the EFIT equilibrium reconstruction when constrained by MSE data with the TRANSP evolved equivalent. It should be noted that the comparison of $q$ profile differences from distinct codes is more difficult than the corresponding iota-bar profile comparison due to the strongly non-uniform (heteroscedastic) nature of the error propagation in $q$: For a given uncertainty in the MSE measurement $\Delta \gamma$, the uncertainty in $q$ at the location of the MSE channel scales as $\Delta q \propto q^2 \Delta \gamma$, i.e. as the square of the $q$ value. In contrast, the uncertainty in iota-bar at
5.3 Analysis of Results

Figure 5.11: Profile (database average) of root mean square difference between TRANSP and EFIT+MSE iota-bar profiles calculated using the NCLASS (full) and Spitzer (dashes) resistivity models.

A fixed location is independent of the value of iota-bar [98]. The rms difference of the TRANSP iota profile when compared to the corresponding EFIT-MSE iota-bar profile averaged over all available pulses is shown in figure 5.11. The rms difference is at a minimum at the plasma boundary as expected; information from the external magnetics ensure $\tau$ is well defined here. As is clear in the figure, the average rms difference calculated by the neoclassical model is poor near the magnetic axis (in the radial range $2.8m < R < 3.2m$). Due to the limit on the safety factor at the magnetic axis in TRANSP, the code is unable to replicate the EFIT calculation of iota-bar in cases of extreme reversed shear in this range. The Spitzer model calculation of iota-bar in these cases is fortuitously closer to the EFIT calculation close to the magnetic axis than the neoclassical model due to the lower resistivity, and hence more slowly evolving iota-bar profile that it calculates.

Neoclassical theory assumes that the trapped ion orbit width is much smaller than the distance $r$ from the magnetic axis [99]. Ion orbits have a width $r_b$ of the order

$$r_b = 2\sqrt{e v / \Omega_\theta}$$  \hspace{1cm} (5.8)

where $\Omega_\theta = \frac{e B_\theta}{m}$ is the poloidal gyrofrequency, $e$ is the electronic charge, $m$...
is the mass of the ion, $B_\theta$ is the poloidal magnetic field, $v$ is the velocity and $\epsilon = \frac{r}{R}$ is the inverse aspect ratio. Inside the region of low $B_\theta$ around the magnetic axis that characterises a contracting current hole, the poloidal gyrofrequency is small and the trapped ion orbit width is of the order of $r$; the ions follow so called potato orbits. In this region, the bootstrap current as calculated from conventional neoclassical theory, is invalid. Since $r_b \propto \sqrt{m}$, electrons follow potato orbits over a much smaller region than the ions in near current hole conditions. Therefore, it is expected that neoclassical resistivity (which depends on electron transport) is relatively unaffected for the contracting current holes studied here. It is clear from figure 5.12 that the mean rmse (averaged over the database) between the TRANSP prediction of the pitch angle and the MSE measurements is higher in the case of current hole than non-current hole discharges for MSE channels #9 – #19 with one exception. This increase may be due to the invalidity of the neoclassical bootstrap current model in the current hole region.

Figure 5.12: Profile (database average) of root mean square difference between TRANSP predicted pitch angle and the MSE measurements for current hole (circles) and non-current hole discharges (squares)
Figure 5.13: The location of the TRANSP $q=3/2$ surface using neoclassical resistivity (dashes) compared to that inferred by MHD (solid).

### 5.3.3 Consistency of TRANSP output with MHD events

As a separate test of the predictive powers of the TRANSP code with respect to the $q$-profile, a comparison with MHD events was conducted. The apparent frequency of neoclassical tearing modes, measured by magnetics sensors, is $n$ times the local bulk plasma rotation frequency (where $n$ is the toroidal mode number of the NTM), with some small offset corresponding to the island rotation in the $E \times B$ frame of reference. Thus, the location of neoclassical tearing modes can be calculated using the rotation frequency profile derived from charge exchange Doppler spectroscopy [100]. This diagnostic provides rotation frequency measurements of the impurity species (carbon) which can differ substantially from that of the bulk plasma [101]. However, for the pulses under investigation here this effect on the rotation frequency was limited to 5%. This information gives a local value of the $q$-profile to be compared with the TRANSP calculation. This comparison was carried out on 10 pulses. In all cases the current profile was evolved using the NCLASS model. On average, the rmse of the TRANSP prediction of the position of the $q = \frac{3}{2}$ surface, when compared with the MHD inferred location of that surface, was 4cm. For example, in figure 5.13 the rmse of
the TRANSP prediction, over the 2.5s of the existence of the NTM, is just 1.4 cm. The variance in the CXS channel location, which contributes to the uncertainty in the MHD inferred location, was 3cm [102].

Figure 5.14: Comparison of the Faraday rotation angle calculated by TRANSP (dotted line) and EFIT+MSE (symbols) with that measured by polarimetry (solid) for shot 52664 at Faraday channel no.2. One standard deviation in the polarimetry data is indicated by the error bar. The hiatus in the EFIT prediction in the interval 4.6 − 5.8s is due to an absence of reliable MSE data.

5.3.4 Consistency of TRANSP output with measured Faraday rotation angle

The consistency of the TRANSP output with polarimetric measurements of Faraday rotation was also examined. The multichannel polarimetry diagnostic at JET shares its 4 vertical and 4 lateral lines of sight (see figure 3.6) with the Far Infra-Red (FIR) interferometer and measures the line-integrated product of the electron density and the parallel component of the poloidal magnetic field. The consistency test was carried out using 18 of the 24 TRANSP runs for which Faraday rotation polarimetry data was available. For the purposes of comparison, the error in the Faraday rotation angle predicted by EFIT, when constrained by MSE pitch angle data (the Faraday rotation data played no part in the equilibrium reconstruction) was
also calculated. Both the TRANSP and EFIT Faraday angle calculations were normalised to agree with the data at the first time point in the EFIT run. At each polarimetry channel it was found that the TRANSP calculation of the Faraday angle was within two standard deviations ($2 \times \sigma$) of the data. The percentage variation in the rmse (calculated as in eqn. 5.7) due to the neoclassical model used in the TRANSP run, when averaged over all the pulses and polarimetry channels, was 6\% (the Hirshmann model again having the highest rmse). The total mean squared error of the TRANSP and EFIT Faraday rotation predictions were calculated using the following equation

$$\text{totalmse}_\alpha = \sum N_{\text{pulse}} \sum N_{\text{channel}} \sum N_{\text{time}} (\alpha_{\text{DATA}} - \alpha_{\text{CAL}})^2$$  (5.9)

where $\alpha_{\text{DATA}}$, $\alpha_{\text{CAL}}$ are the measured and predicted Faraday rotation angles respectively. $N_{\text{pulse}}$, $N_{\text{channel}}$, and $N_{\text{time}}$ are the number of pulses, polarimetry channels and time points to be summed over. The ratio of the TRANSP total mean squared error to that of EFIT was 1.007 when NCLASS was used to model the resistivity and bootstrap current in TRANSP but this ratio rose to 1.14 when Spitzer resistivity was used. This consistency of the TRANSP Faraday rotation calculation with both the polarimetry data and the EFIT calculation is demonstrated in figure 5.14.

### 5.3.5 Consistency of TRANSP calculated pressure profile and MSE constrained equilibrium fits

It is of interest to investigate what the effect on the EFIT+MSE results would be if the TRANSP calculated pressure profile (including fast particle contribution) is added as an additional internal constraint on the equilibrium solution [103]. Such a comparison would, in effect, further elucidate the level of consistency between the EFIT and TRANSP codes. To begin, a well diagnosed optimised shear discharge was selected (pulse #47413) for analysis. TRANSP was run without any input from the MSE pitch angle data and the self-consistency of the output was checked by comparing the loop voltage and total neutron rate calculated with the measurements. EFIT was then run at several time points for each of the following cases: (i) including the TRANSP total pressure profile as a constraint, (ii) including the MSE pitch angle as a constraint, and (iii) with both constraints included. In all three cases EFIT was constrained by the magnetic sensor data as normal.
5.3 Analysis of Results

The parameterisation of the EFIT current profile was chosen to facilitate good agreement with both the pressure profile and the pitch angle data. A curvature penalty was also imposed to ensure a realistically shaped current profile.

In figure 5.15 the pressure profile calculated by EFIT, for each of the three cases outlined above, is compared with the TRANSP calculation. The peakedness of the TRANSP pressure profile is a consequence of the noise associated with the Monte Carlo calculation used to track the fast particles in the code. When this data is input in the EFIT equilibrium calculation, as the only internal constraint on the current profile, a consistent pressure profile is output (aside from the peakedness near the magnetic axis). From the figure it is evident that when MSE pitch angle data is added in addition as an internal constraint, the EFIT calculated pressure profile is also consistent with that input.

By the same token, the EFIT calculated pitch angle is consistent (over the reliable MSE channels) with the input MSE pitch angle data even when the TRANSP pressure also exerts an influence on the equilibrium calculation (figure 5.16). The flexibility of EFIT, evident in its ability to find an equilibrium solution in agreement both with the TRANSP pressure and MSE pitch

Figure 5.15: TRANSP pressure profile and profiles calculated by EFIT with (i) pressure constraint only (EFIT-PRESS) (ii) MSE constraint only (EFIT-MSE) and (iii) MSE and pressure constraint (EFIT-MSE+PRESS)
angle profiles, can also be seen in its calculation of the plasma current profile (figure 5.17). This flexibility is due to the degree of freedom in the current profile calculation allowed by the unconstrained \( FF' \) term (see figure 5.18) even when the pressure term is constrained by measurements (see equation 2.26). When the TRANSP pressure profile alone (apart from the external magnetic measurements) is used as a constraint, a peaked current profile results. In the cases where MSE alone or together with the pressure profile is included a optimised shear style hollow current profile results.

5.4 Discussion

The evolution of the pitch angle in JET as calculated by TRANSP was shown to be consistent with that measured by the motional Stark effect diagnostic when a neoclassical model of the resistivity and bootstrap current was employed. This result is in agreement with earlier investigations of the resistivity model at JET [84]- [86] and at other tokamaks such as TFTR [62], [87], JT60 [83] and DIII-D [88]. For instance, at TFTR the neoclassical calculation was found to be within two standard deviations of the data in most cases, while the classical model calculation was grossly incorrect [87]. The
consistency of the predicted MSE signals from TRANSP was demonstrated in a comparison of the TRANSP evolved current profile with that found by the equilibrium code EFIT when the latter was constrained by the MSE pitch angle data. This agreement was found to hold over a large range of plasma heating scenarios. However, agreement with the MSE pitch angle data was found to be worse, on average, in the case of current hole discharges, possibly a consequence of the invalidity of the neoclassical bootstrap current calculation in the current hole region. In order to investigate this problem, numerical calculations of the bootstrap current in this region have been carried out [104]. The Spitzer model, as expected, was significantly less accurate than the neoclassical models in predicting the pitch angle evolution. It was found that this inaccuracy was for the most part due to the limitations of the Spitzer resistivity rather than the absence of a bootstrap current in this model. The Hirshmann model, valid only at low inverse aspect ratio, was found to be systematically less accurate in its prediction of the MSE pitch angle and Faraday rotation angle than the NCLASS and Sauter models. No dependence on neutral beam or ICRF heating was found. The location of a specific rational surface in the TRANSP evolved $q$-profile was found to be consistent with MHD data. The location of neoclassical tearing modes, and
5.4 Discussion

Figure 5.18: $FF'$ profiles, scaled by $10^{-6}$, calculated by EFIT with (i) MSE and pressure constraints and (ii) MSE constraint only and (iii) pressure constraint only. $\rho$ is the normalised radius $\frac{r}{a}$.

hence the $q = \frac{3}{2}$ surface, was found by comparing their rotation rates with the measurements of charge exchange spectroscopy. Due to the accuracy with which TRANSP predicts the location of NTM modes, we conjecture that the effect of the modes on the magnetic field diffusion is localised. The consistency of TRANSP with other types of MHD events such as sawtooth crashes could also be examined. Finally, the evolution of the Faraday rotation angle calculated by TRANSP was found to be consistent with the polarimetry data, and also consistent with the EFIT results when the equilibrium calculation was constrained by MSE pitch angle data.

In summation, it has been shown that TRANSP, given an accurate initial pitch angle profile is accurate in its evolution of that quantity, and its derivative the current profile, thereafter. Potentially, this should enable the current profile to be calculated with (modest) uncertainties typified by the results presented here even in the absence of MSE data in the following situations: (i) After a period of availability early in the discharge there is no further usable MSE data due to the deployment of an unfavourable combination of NBI sources, etc. (ii) Assuming a standard pre-heating scenario with a reproducible early current profile evolution as verified from discharges with MSE data, it becomes feasible to use TRANSP to evolve the current profile for
discharges with the same early phase (thus providing a reliable initial condition) where no MSE data are available. Extending this work to include those pulses with LHCD during the main heating phase would markedly increase the range of pulses that could be modelled using the technique presented here and work is in progress to achieve this.

5.5 Thesis conclusion

This thesis is concerned with comparing the current density profile found from resistive diffusion with that found from MHD equilibrium calculations. The theory that underpins these two approaches to current profile recovery is explored in chapter 2. The magnetic field diffusion equation in cylindrical geometry (eqn. 2.55) was investigated in particular detail leading to an equation which describes the evolution of $B_\theta$ in the case of uniform resistivity (or magnetic diffusivity $\lambda_m$) across the plasma (eqn. 2.76). Solutions to the diffusion equation in cases of simple non-uniform diffusivity, namely in some monomial $\lambda_{mono}(r) = \lambda_o r^N$ and modified monomial $\lambda_{mono}(r) = \lambda_o(1 + g^N r^N)$ cases, were also found (section 4.7.2). In order to study the more general case of plasmas with more complex non-uniform resistivity profiles, the magnetic field diffusion equation was numerically solved using Mathematica (section 4.7). In general, a non-linear steady state $B_\theta$ profile was found (where the boundary values of the $B_\theta$ profile are fixed). This steady state profile was found to be consistent with that calculated by TRANSP (section 4.7.5). Furthermore it was established that the evolution of the magnetic field can be described by a single quantity, an effective magnetic diffusivity $\lambda_{eff}$, despite the local resistivity across the plasma being non-uniform. This was supported by the Mathematica, TRANSP and analytical results. Effective diffusivity models were developed that allow $\lambda_{eff}$ to be calculated for a given set of electron temperature, density and Z-effective data. Two of these models are based on different methods of averaging the temperature profile and provide accurate predictions of $\lambda_{eff}$. A third, less accurate, model is based on the analytic solutions.

In chapter 3, the EFIT equilibrium code was described. In particular a study of the iron model in EFIT was undertaken. Since no measurements of the magnetic field at the iron exists, a model is needed to calculate the surface currents there (since EFIT is a free-boundary code, all current sources
must be included). Using data from dry runs (in which all other currents are well determined), the magnetic field predictions of the model were found to be inaccurate. In section 3.5 the effect of modifying the iron model on the EFIT equilibrium solution was investigated. It was found that EFIT is relatively robust to these changes, reflecting the following fact: given a sufficient number of magnetic measurements, a full-domain equilibrium solver like EFIT can accommodate unknown currents by allowing the magnetic field at the probe locations to vary within their experimental uncertainty when solving the Grad-Shafranov equation.

In the final chapter, the central question of this thesis was answered; whether the current profile determined by resistive diffusion is consistent with MHD equilibrium results. In the first instance, it was found that the pitch angle predicted by TRANSP via field diffusion is consistent with the MSE measurements. Agreement was also found with Faraday rotation and MHD data. It was also found that the current profile (and $\iota$ profile) calculated by TRANSP and EFIT are in broad agreement. In addition, it was confirmed that a neoclassical model of resistivity applies to JET plasmas.

Future research suggested by this thesis include

- Extending the effective diffusivity model to include cases of time varying $T_e, n_e$ and $Z_{\text{eff}}$ profiles. Allowing the magnetic field at the plasma boundary to vary is also desirable.

- Concluding a more thorough test of the effective diffusivity model in the case of neoclassical resistivity. The model should also be tested on TRANSP output over a wider parameter space.

- Modifying the EFIT code to be able to calculate a self-consistent equilibrium in the case of current hole plasmas. In order to improve EFIT convergence with the MSE data various approaches could be adopted. The first involves using alternatives to $\psi$ to label the magnetic surfaces in the basis functions to better approximate the current hole. For example, the equilibrium code ESC uses the toroidal flux function $\phi$. Other approaches involve retaining $\psi$ as the main coordinate in solving the G-S equation but attempting to find a solution which approximate a current hole as closely as possible.

- Extending the work of chapter 5 to include cases where there is LHCD during the main heating phase. This work would allow a more thorough
investigation of the resistive diffusion of the magnetic field and current density in current hole plasmas. It is also of interest to test the validity of the neoclassical bootstrap current inside the current hole radius.
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